

## B.4 Solutions to Exam MFE/3F, Spring 2009

The questions for this exam may be downloaded from

<http://www.soa.org/files/pdf/edu-2009-05-mfe-exam.pdf>

1. [Section 4.2] The multipliers for up and down moves and the risk-neutral probability are

$$u = e^{(r-\delta+0.5\sigma)} = e^{0.05-0.05+0.3} = e^{0.3} = 1.34986$$

$$d = e^{-0.3} = 0.74081$$

$$p^* = \frac{1}{1+e^\sigma} = \frac{1}{1+e^{0.3}} = 0.425557$$

where we've used equation (3.6) to calculate  $p^*$ , since this tree is the forward tree. The resulting stock prices are shown in Figure B.3.

At the ending nodes, the option only pays off at the highest node. Pulling back one year:

$$C_u^{\text{tentative}} = e^{-0.05}(0.425557)(82.2119) = 33.2796$$

However, 33.2796 is less than the exercise value of 34.9859, so  $C_u = 34.9859$ . Then

$$C = e^{-0.05}(0.425557)(34.9859) = \boxed{14.1623} \quad (\text{E})$$

2. [Sections 13.1 and 13.2] The average of the 12 stock prices is

$$\frac{105 + 120 + \cdots + 110 + 115}{12} = 110$$

so the Asian call option pays 10.

The up-and-out pays nothing since the 125 barrier was hit; the up-and-in pays 5 since the barrier of 120 was hit and  $115 - 110 = 5$ . The answer is (B).

3. [Lesson 3] The risk-neutral probability of an up is

$$p^* = \frac{e^{r-\delta} - d}{u - d} = \frac{e^{0.05-0.10} - 0.8}{1.1 - 0.8} = 0.504098$$

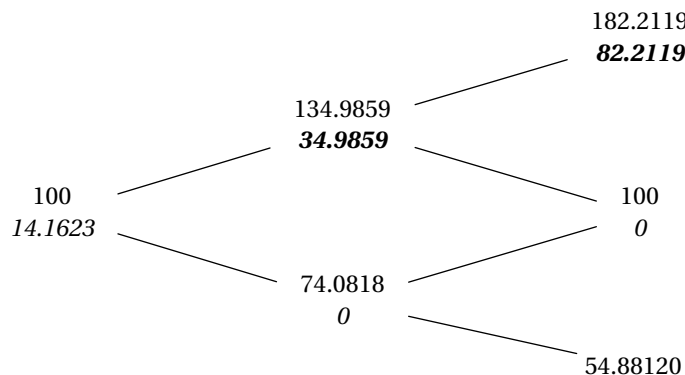


Figure B.3: Binomial tree for S09:1

so the value of the call is  $e^{-0.05}(0.504098)(55 - 50) = 2.3976 > 1.90$ , so buy the option and sell shares of WWW. When buying a call, shares must be sold since a call results in buying stock if it pays off. Lend the extra money. This is choice **(B)**.

The official solution also works this out with the replicating portfolio.

4. [Section 14.1] As listed in Table 14.1, the appropriate formula for this asset-or-nothing put option  $S | S < K$  is  $Se^{-\delta t}N(-d_1)$ .

$$d_1 = \frac{\ln(1/0.6) + 0.025 - 0.02 + 0.5(0.2^2)}{0.2} = 2.67913$$

$$N(-d_1) = 0.00369$$

$$S | S < K = 1000e^{-0.02}(0.00369) = 3.61693$$

Multiplying by one million, the answer is **(D)**.

5. [Section 24.1] The zero-coupon bond is worth  $e^{-0.18} = 0.83527$  at the top node,  $e^{-0.12} = 0.88692$  at the two middle nodes, and  $e^{-0.06} = 0.94176$  at the bottom node, so the option pays off at the top 3 nodes. Discounting with probabilities to the initial node:

$$P = (0.7^2)e^{-0.12-0.15}(0.9 - 0.83527) + (0.7)(0.3)e^{-0.12-0.15}(0.9 - 0.88692) + (0.7)(0.3)e^{-0.12-0.09}(0.9 - 0.88692)$$

$$= 0.02421 + 0.00210 + 0.00222 = \boxed{0.029} \quad \text{(E)}$$

6. [Lesson 18] The derivatives of  $Y(t)$  are

$$Y_X = -\frac{1}{X^2} \qquad Y_{XX} = \frac{2}{X^3} \qquad Y_t = 0$$

By Itô's Lemma,

$$dY(t) = Y_X dX + 0.5Y_{XX}(dX)^2 + Y_t dt$$

$$= -\frac{8 - 2X(t)}{X(t)^2} dt - \frac{8}{X(t)^2} dZ(t) + \frac{64}{X(t)^3} dt$$

with the last term resulting from  $(dX(t))^2 = (8 dZ(t))^2 = 64 dt$ . Ignore the  $dZ(t)$  term, since we don't need that for this question. The coefficient of  $dt$  is

$$-Y(t)^2 \left( 8 - \frac{2}{Y(t)} \right) + 64Y(t)^3 = -8Y(t)^2 + 2Y(t) + 64Y(t)^3$$

so  $\alpha(y) = -8y^2 + 2y + 64y^3$ , and  $\alpha(1/2) = -8(0.25) + 2(0.5) + 64(0.125) = \boxed{7}$ . **(D)**

7. [Lesson 5] Tricky question: even though probabilities, assumptions, and payoffs don't change, the answer is not A because the risk-neutral probability changes.

The second method in the official solution is easier, and that is what I will use here.

From the information given, \$1.13 is  $2p^*$  discounted one year, or

$$1.13 = 2 \left( \frac{e^r - d}{u - d} \right) e^{-r} = 2 \left( \frac{e^r - 0.8}{0.4} \right) e^{-r} = 5 - 4e^{-r}$$

$$e^{-r} = \frac{5 - 1.13}{4} = 0.9675$$

In the revised tree,

$$C = 2 \left( \frac{e^r - 0.6}{0.6} \right) e^{-r} = \frac{10}{3} - 2(0.9675) = \boxed{1.3983} \quad \text{(D)}$$

8. [Lesson 19] By the Black-Scholes Equation (19.1),

$$0.5S^2\sigma^2 V_{SS} + V_t + V_S S(r - \delta) = rV$$

For  $V = e^{rt} \ln(S(t))$ ,

$$\begin{aligned} V_S &= \frac{e^{rt}}{S(t)} & V_{SS} &= -\frac{e^{rt}}{S(t)^2} & V_t &= rV \\ -0.5e^{rt}\sigma^2 + (r - \delta) + rV &= rV \\ -0.5\sigma^2 + r - \delta &= 0 \\ \delta &= r - 0.5\sigma^2 \end{aligned}$$

Plugging in the values we're given,  $\delta = 0.055 - 0.5(0.3^2) = \boxed{0.01}$ . (B)

The official solution has the following alternative. The risk-neutral expectation of the prepaid forward price of the security must equal the current value of the security. The current value ( $t = 0$ ) of the security is  $\ln(S(0))$ , while the prepaid forward price of it is  $e^{-rt}$  times its price, or  $\ln(S(t))$ . So  $\ln(S(0)) = \mathbf{E}[\ln(S(t))]$  for all  $t$ . But  $\ln(S(t))$  has a normal distribution with mean  $r - \delta - 0.5\sigma^2$  under the risk-neutral measure, so  $r - \delta - 0.5\sigma^2 = 0$ .

9. [Subsection 1.2.5] This question requires a currency translation and put-call parity.

If you do put-call parity first, then we calculate the value of a four-year dollar-denominated European call option on yen with strike price \$0.008:

$$\begin{aligned} 0.0005 - C &= 0.008e^{-4(0.03)} - 0.011e^{-4(0.015)} = -0.00326405 \\ C &= 0.0005 + 0.00326405 = 0.00376405 \end{aligned}$$

Then multiply by 125 since we need a put on 1 dollar or 125(0.008) dollars, and divide by 0.11 to translate the currency into dollars:  $\yen0.00376405(125/0.11) = \boxed{\yen42.7733}$ . (E)

If you want to do the currency translation first: a four-year put option to pay \$0.008 and get ¥1 is multiplied by 125 to make it pay \$1 and get ¥125, and then the price is \$125(0.0005). The price in yen is \$125(0.0005)/0.11 = ¥5.6818. By put-call parity, a put option on dollars to pay ¥125 and get \$1 is worth:

$$\begin{aligned} P - 5.6818 &= 125e^{-4(0.015)} - (1/0.11)e^{-4(0.03)} = 37.0914 \\ P &= 5.6818 + 37.0914 = \boxed{42.7733} \end{aligned}$$

10. [Section 20.2] As discussed in Example 20D, a risk-free portfolio is obtained by zeroing out the volatility, so if  $x_i$  is the amount to invest in asset  $i$  in this question, then

$$0.2x_1 - 0.25x_2 = 0$$

which implies  $x_1 = (5/4)x_2$ . Since  $x_1 + x_2 = 1000$ , it follows that  $x_1 = (5/9)(1000) = \boxed{555.56}$ . (C)

11. [Lesson 22] The expected value of  $S(1)^a$  is calculated from formula (22.1):

$$\mathbf{E}[S(1)^a] = S(0)^a e^{a(a-\delta)+0.5a(a-1)\sigma^2}$$

and is equal to 1.4. We are being asked for the prepaid forward price of  $S(1)^a$ , or formula (22.3):

$$F_{0,1}^P(S(1)^a) = S(0)^a e^{-r+a(r-\delta)+0.5a(a-1)\sigma^2}$$

The quotient of the second formula over the first is

$$\frac{F_{0,1}^P(S^a)}{\mathbf{E}[S(1)^a]} = e^{-r-a(\alpha-r)}$$

so we need to back out  $a$ . The stochastic differential equation for  $S(t)$  has  $\alpha = 0.05$  and  $\sigma = 0.2$ .

$$\begin{aligned}\mathbf{E}[S(1)^a] &= S(0)^a e^{a(\alpha-\delta)+0.5a(a-1)\sigma^2} \\ \ln 1.4 &= a \ln 0.5 + 0.3 + (0.05a + 0.02a(a-1)) \\ 0.02a^2 + (\ln 0.5 + 0.03)a - \ln 1.4 &= 0 \\ a &= -0.49985\end{aligned}$$

The other solution to the quadratic is rejected since it is positive. Then, with  $r = 0.03$ ,

$$F_{0,1}^P(S^a) = 1.4e^{-0.03+0.49985(0.05-0.03)} = \boxed{1.372} \quad (\text{C})$$

**12. [Subsection 1.2.1 and Lesson 2]**

I.  $C(50, T)$  is worth more than  $C(55, T)$  since calls decrease in value with increasing strike prices, and the difference in values is less than  $5e^{-rT}$  because if you buy  $C(50, T)$  and sell  $C(55, T)$ , the most you can get is 5 at time  $T$ , which is worth  $5e^{-rT}$  at time 0. ✓

II. By put-call parity,

$$P(50, T) - C(50, T) = 50e^{-rT} - S$$

and  $P(45, T) \leq P(50, T)$ , so  $P(45, T) - C(50, T) \leq 50e^{-rT}$ , and is certainly also less than  $55e^{-rT}$ , but the left side inequality is incorrect. ✗

III. We proved the right hand inequality in (II). The left hand inequality follows from

$$P(45, T) - C(45, T) = 45e^{-rT} - S$$

and  $C(45, T) \geq C(50, T)$ . ✓ (E)

The answer choices are asymmetric, something which was not allowed on previous exams, but II and III cannot both be true, so they perhaps were forced to make them asymmetric.

**13. [Section 9.1]** By the Black-Scholes formula, the value of the option at time 8 months, with 4 months left to expiry, is

$$d_1 = \frac{\ln(85/75) + (0.05 + 0.5(0.26^2))(1/3)}{0.26\sqrt{1/3}} = 1.01989$$

$$d_2 = 1.0199 - 0.26\sqrt{1/3} = 0.86978$$

$$N(d_1) = N(1.01989) = 0.84611$$

$$N(d_2) = N(0.86978) = 0.80779$$

$$C = 85(0.84611) - 75e^{-0.05(1/3)}(0.80779) = 12.3365$$

The eight-month holding profit, taking into account the eight-month interest cost on the original investment of 8, is  $12.3365 - 8e^{0.05(2/3)} = \boxed{4.0653}$ . Using the rounding rules for the normal CDF in effect for this exam, the answer would've been 406, or (A).

**14. [Subsection 24.2.2]** First, in a Black-Derman-Toy tree, the vertical ratios between interest rates are constant, so  $0.8/r_{ud} = r_{ud}/0.2$  and therefore  $r_{ud} = 0.4$ .

By formula (24.1), the forward's price is  $P(0,3)/P(0,2)$ . We calculate these in terms of the initial interest rate  $r_0$ . In the following,  $P_u(1,3)$  is the price of a two-year bond at the upper node after one year, and  $P_d(1,3)$  is the price of a two-year bond at the lower node after one year.

$$\begin{aligned} P(0,2) &= \frac{0.5}{1+r_0} \left( \frac{1}{1.6} + \frac{1}{1.3} \right) = \frac{0.697115}{1+r_0} \\ P_u(1,3) &= \frac{0.5}{1.6} \left( \frac{1}{1.8} + \frac{1}{1.4} \right) = 0.396825 \\ P_d(1,3) &= \frac{0.5}{1.3} \left( \frac{1}{1.4} + \frac{1}{1.2} \right) = 0.595238 \\ P(0,3) &= \frac{0.5}{1+r_0} (0.396825 + 0.595238) = \frac{0.496032}{1+r_0} \end{aligned}$$

The answer to the question is  $1000F_{0,2}(P(2,3)) = 1000(0.496032/0.697115) = \boxed{711.55}$ . (E)

The question omitted  $r$  and used high interest rates to try to trick you. Some students mistakenly thought that the forward price is the probability-weighted average of the one-year bond prices at the end of two years, or

$$0.25 \left( \frac{1}{1.8} \right) + 0.5 \left( \frac{1}{1.4} \right) + 0.25 \left( \frac{1}{1.2} \right) = 0.70437$$

and 704.37 is not one of the five answer choices.

**15. [Section 26.3]** We recognize the model as a Vasicek model with  $a = 0.1$  and  $\sigma = 0.05$ . The Sharpe ratio is deduced by comparing the risk-neutral process to the true process; the Sharpe ratio  $\phi$  times  $\sigma dt$  is added to go from the latter to the former, and the risk-neutral process is  $0.005 dt$  more than the true process, so  $\phi = 0.005/0.05 = 0.1$ . Then

$$\begin{aligned} B(t, T) &= \frac{1 - e^{-a(T-t)}}{a} \\ B(2, 5) &= \frac{1 - e^{-0.1(3)}}{0.1} = 2.59182 \\ q(0.04, 2, 5) &= B(2, 5)\sigma = 2.59182(0.05) \\ \frac{\alpha(0.04, 2, 5) - 0.04}{q(0.04, 2, 5)} &= 0.1 \\ \alpha(0.04, 2, 5) &= 2.59182(0.05)(0.1) + 0.04 = \boxed{0.05296} \quad (\text{C}) \end{aligned}$$

**16. [Lesson 16]** The parameters  $m$  and  $\nu$  of the lognormal distribution of the stock price after 9 months are

$$\begin{aligned} m &= (\alpha - \delta - 0.5\sigma^2)(t) = (0.1 - 0.5(0.3^2))(0.75) = 0.04125 \\ \nu &= \sigma\sqrt{t} = 0.3\sqrt{0.75} = 0.25981 \end{aligned}$$

So the probability that the stock price is more than 125 is

$$1 - N\left(\frac{\ln(125/100) - 0.04125}{0.25981}\right) = 1 - N(0.70010) = \boxed{0.24193} \quad (\text{A})$$

17. [Subsection 10.1.1] The formula for a put's delta is  $e^{-\delta t} (N(d_1) - 1)$  (equation (10.3)), and since  $N(-d_1) = 1 - N(d_1)$ , this is the same as  $-e^{-\delta t} N(-d_1)$ . Since  $\delta = 0$ ,

$$\begin{aligned} -N(-d_1) &= -0.4364 \\ d_1 &= 0.16010 \end{aligned}$$

We'll use  $d_1 = 0.16$ , in accordance with the rounding rules for the normal distribution in effect for this exam. The answer is not significantly different if 0.1601 is used.

We set up the quadratic equation for  $\sigma$ .

$$\begin{aligned} \frac{r + 0.5\sigma^2}{\sigma} &= 0.16 \\ 0.012 + 0.5\sigma^2 &= 0.16\sigma \\ 0.5\sigma^2 - 0.16\sigma + 0.012 &= 0 \end{aligned}$$

The two solutions are  $\sigma = 0.12, 0.20$ . Higher volatility leads to higher put prices, so if 20% satisfies (i), 12% certainly does, and since the answer is unique, we know the answer has to be **12%**. (A) The official solution shows how you can verify  $\sigma < 0.14$  without explicitly valuing the put with Black-Scholes formula for  $\sigma = 0.2$ ; you can express the Black-Scholes formula for the put over the stock price in terms of  $N(d_1)$  and  $N(d_2)$ , set  $d_2 = d_1 - \sigma$ , and then get an upper bound for  $\sigma$ .

18. [Lesson 20] The Sharpe ratios of the two stocks must equal. Let  $\alpha_i$  and  $\sigma_i$  be the rates of return and volatilities of the stocks with prices  $S_i$ . Then

$$\begin{aligned} \alpha_1 &= 0.1 + 0.5(0.2^2) = 0.12 \\ \alpha_2 &= 0.125 + 0.5(0.3^2) = 0.17 \\ \frac{0.12 - r}{0.2} &= \frac{0.17 - r}{0.3} \\ 0.036 - 0.3r &= 0.034 - 0.2r \\ 0.1r &= 0.002 \\ r &= \mathbf{0.02} \quad (\text{A}) \end{aligned}$$

19. [Section 9.1] The notation  $\text{Var}(\ln F_{t,1}^P(S))$  looks a little weird, but it denotes the variance of the prepaid forward from time  $t$  to time 1, as of time 0, as a function of  $t$ . When  $t = 0$ , there is no volatility since the prepaid forward price is known, but as time goes on, the volatility, from the perspective of valuation at time 0, keeps growing. So  $\sigma^2 = 0.01$  and  $\sigma = 0.1$ . We can now use the Black-Scholes formula on the prepaid forward of the stock, whose value at time 0 is

$$50 - 5e^{-rt} = 50 - 5e^{-0.75(0.12)} = 45.43034$$

Plugging into the prepaid forward version of the Black-Scholes formula, equation (9.1),

$$\begin{aligned} d_1 &= \frac{\ln(45.43034/45e^{-0.12}) + 0.5(0.1^2)}{0.1} = 1.34518 \\ d_2 &= 1.34518 - 0.1 = 1.24518 \end{aligned}$$

Using the rounding rules in effect at this exam,

$$\begin{aligned} N(-d_1) &= N(-1.35) = 0.0885 \\ N(-d_2) &= N(-1.25) = 0.1056 \end{aligned}$$

$$P = 45e^{-0.12}(0.1056) - 45.43034(0.0885) = 0.1941$$

and 100 units have value **19.41**. (D) Using 5-decimal precision for the normal CDF,

$$N(-d_1) = N(-1.34518) = 0.08928$$

$$N(-d_2) = N(-1.24518) = 0.10653$$

$$P = 45e^{-0.12}(0.10653) - 45.43024(0.08928) = 0.19575$$

and 100 units have value **19.58**.

20. [Section 12.2] The delta-gamma approximation is equation (12.2) without  $h\theta$ . Thus we have

$$\Delta\epsilon + 0.5\Gamma\epsilon^2 = 2.21 - 2.34 = -0.13$$

$$-0.181\epsilon + 0.5(0.035)\epsilon^2 = -0.13$$

$$0.0175\epsilon^2 - 0.181\epsilon + 0.13 = 0$$

$$\epsilon = \frac{0.181 \pm \sqrt{0.181^2 - 4(0.0175)(0.13)}}{0.035} = 9.5663, 0.7765$$

The original stock price is  $86 - \epsilon$ . Using  $\epsilon = 0.7765$  gets an original stock price of  $S(0) = \mathbf{85.2235}$ . (C). Using 9.5663 gets a stock price below 80, violating (i). The delta-gamma approximation is invalid for such a large change, since the squared epsilon in the gamma term overwhelms delta in such a case.

## B.5 Solutions to Sample Questions

The SOA sample questions and solutions for Exam MFE/3F can be downloaded from

[http://www.beanactuary.org/exams/pdf/MFE\\_SampleQS1-76.pdf](http://www.beanactuary.org/exams/pdf/MFE_SampleQS1-76.pdf)

The proportion of these questions that are based on difficult topics such as Brownian motion is greater than the proportion of questions that a real exam would have. The solutions are lengthy and often go both beyond the question and the McDonald textbook; they tend to avoid using short cuts and instead derive everything from first principles, which makes them long. They are often not the way you would work out the question on an exam.

The questions and their solutions are more an educational tool than a sample exam, and in fact, rather than providing the questions and solutions separately, each question is followed by its solution, which makes it hard to use these questions as a sample exam.

The solutions provided here are the way I would solve these questions if they came up on an exam. I usually present only one method, my favorite method.

1. [Subsection 1.2.1] This is a straightforward put-call parity question. By put-call parity

$$\begin{aligned} P(S, K, t) - C(S, K, t) &= Ke^{-rt} - Se^{-\delta t} \\ -0.15 &= 70e^{-4r} - 60 \\ e^{-4r} &= \frac{60 - 0.15}{70} = 0.855 \\ r &= -\frac{\ln 0.855}{4} = \boxed{0.03916} \quad (\text{A}) \end{aligned}$$

2. [Subsection 2.4.1] The call prices do not satisfy convexity, since  $6 > (1/3)(11) + (2/3)(3) = 5\frac{2}{3}$ . The put prices, however, satisfy convexity.

Mary's portfolio, if two calls with strike price 55 are longed, is a butterfly spread. A butterfly spread is the standard method for exhibiting arbitrage when convexity is violated. The portfolio costs  $(11) + 2(3) - 3(6) = -1$ , and after lending 1, there is no initial cash flow. At expiry, the amount paid by Mary for the three call options shorted will always be no greater than the amount received on the three call options longed, so Mary will gain the repayment of the loan at the very least, with no risk of loss.

In Peter's portfolio, every pair of calls longed and puts shorted with the same strike price will be worth, at expiry,  $S - K$ . If three calls with strike price 50 are shorted and three puts with strike price 50 longed, Peter will receive a net of \$2 initially, since  $2(3 - 11) + (11 - 3) + 3(8 - 6) = -2$ . At expiry, the portfolio will be worth 0, since the pair of 40-strike options will yield  $S - 40$ ; the two pairs of 55-strike options will yield  $S - 110$ ; and the three pairs of 50-strike options will yield  $150 - 3S$ , and these add up to 0. Peter will gain the repayment of the loan.

Thus Mary's and Peter's portfolios exhibit arbitrage. (D)

3. [Section 13.3] The present value of the payout is equated to the premium  $\pi$ . We are given the price of a put that pays the maximum of 0 and  $103 - S$ , so we would like to turn the max expression for the payoff into  $\max(0, 103 - S)$ . We do this by factoring out  $1/S(0)$ :

$$\max\left(S(T)/S(0), (1 + g\%)^T\right) = \frac{\max(S(1), 103)}{100} = \frac{S(1) + \max(0, 103 - S(1))}{100}$$

The present value of  $S(1)$  is  $S(0)$  or 100, since  $S(1)$  incorporates dividends. If it didn't incorporate dividends, the present value would be  $S(0)e^{-\delta}$ , since  $F_{0,T}^P(S) = Se^{-\delta t}$ . (See Table 1.2.) The present value of the maximum expression is therefore

$$\text{PV}\left(\max\left(S(T)/S(0), (1 + g\%)^T\right)\right) = \frac{100 + 15.21}{100} = 1.1521$$

Equating the present value of the payout to the premium,

$$\begin{aligned}\pi \times (1 - y\%) \times 1.1521 &= \pi \\ 1 - y\% &= \frac{1}{1.1521} = 0.86798\end{aligned}$$

so  $y\% = 1 - 0.86798 = \boxed{0.13202}$ .

**4. [Section 4.1]** Although not needed for the solution,  $u = e^{0.25}$  and  $d = e^{-0.15}$ , so this binomial tree is based on forward prices with  $\sigma = 0.2$ . Thus you can calculate  $p^*$  using the shortcut formula (3.6):  $1/(1 + e^{\sigma\sqrt{h}}) = 1/(1 + e^{0.2}) = 0.4502$  if you wish.

Since the stock pays no dividends, the American call option is worth the same as a European call option, so we have no need to induct on the tree. We just have to calculate the option payoffs at the ending nodes, weight them with risk-neutral probabilities, and discount them. The risk-neutral probability of an up-movement is

$$p^* = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502$$

The stock prices at the three ending nodes are  $S_{uu} = 20(1.284^2) = 32.97$ ,  $S_{ud} = 20(1.284)(0.8607) = 22.10$ , and  $S_{dd} = 20(0.8607^2) = 14.82$ . The call payoffs from top to bottom are 10.97, 0.10, and 0. Weighting and discounting:

$$C = e^{-0.1} \left( (0.4502^2)(10.97) + 2(0.4502)(0.5498)(0.10) \right) = \boxed{2.06} \quad (\text{C})$$

**5. [Section 4.3]** They didn't tell you which binomial tree to use. They intended that you use one based on forward prices. I think they would be more clear regarding which tree to use on a real exam.

For the tree with forward prices,  $r$  is the domestic dollar interest rate or 8% and  $r_f$  is 9%.

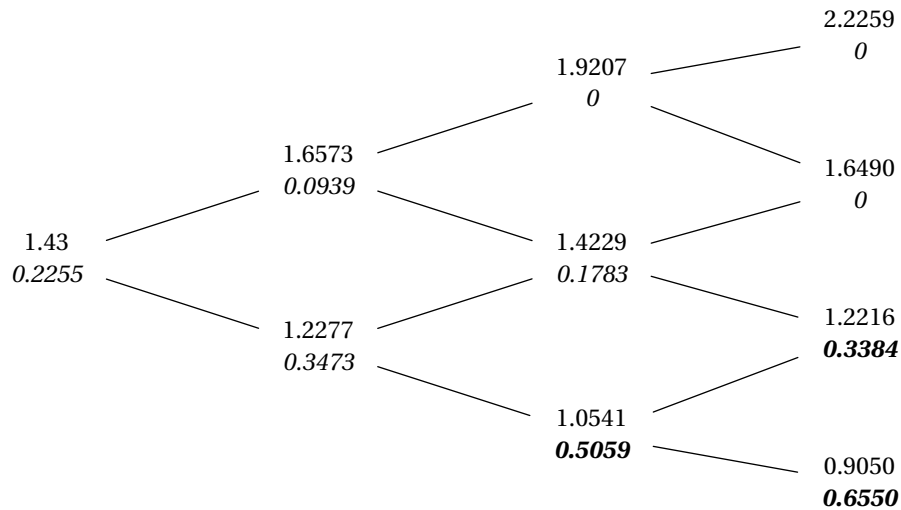
$$\begin{aligned}u &= e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.09)(0.25)+0.3(0.5)} = 1.1589 \\ d &= e^{-0.0025-0.3(0.5)} = 0.8586 \\ p^* &= \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.3(0.5)}} = 0.4626\end{aligned}$$

We then calculate the exchange rates and put payoffs at the ending nodes as indicated in Figure B.4 We compute the put prices as:

$$\begin{aligned}P_{uu} &= 0 \\ P_{ud} &= e^{-0.08(0.25)}(0.5374)(0.3384) = 0.1783 \\ P_{dd}^{\text{tentative}} &= e^{-0.02} \left( 0.4626(0.3384) + 0.5374(0.6550) \right) = 0.4985\end{aligned}$$

However, at the  $dd$  node,  $1.56 - 1.0541 = 0.5059 > 0.4985$ , so the option is exercised early at that node. Continuing,

$$\begin{aligned}P_u &= e^{-0.02}(0.5374)(0.1783) = 0.0939 \\ P_d &= e^{-0.02} \left( 0.4626(0.1783) + 0.5374(0.5059) \right) = 0.3473 \\ P &= e^{-0.02} \left( 0.4626(0.0939) + 0.5374(0.3473) \right) = \boxed{0.2255}\end{aligned}$$



**Figure B.4:** Binomial tree for dollar/pound exchange rate question, sample question 5

6. [Section 9.1] The Black-Scholes formula gives:

$$d_1 = \frac{\ln(20/25) + (0.05 - 0.03 + 0.5(0.24^2))(0.25)}{0.24\sqrt{0.25}} = -1.7579$$

$$d_2 = -1.7579 - 0.24\sqrt{0.25} = -1.8779$$

$$N(d_1) = N(-1.76) = 0.0392$$

$$N(d_2) = N(-1.88) = 0.0301$$

$$C(S, K, \sigma, r, t, \delta) = 20e^{-0.0075}(0.0392) - 25e^{-0.0125}(0.0301) = 0.03499$$

For 100 units, the price is **3.499**. (C)

7. [Section 9.2] The question uses Brownian motion language to state that the Black-Scholes framework applies and to state the volatility, but only requires the Black-Scholes or Garman-Kohlhagen formula.

Since the price of the option is denominated in dollars, the easier way to do this is from the American perspective. Yen must be sold for dollars, so the needed option is a put on yen. The domestic risk-free rate, which is used for  $r$ , is 3.5% and the foreign risk-free rate, which is used for  $\delta$ , is 1.5%. The option is at-the-money, so  $K = x_0$  and  $\ln(x_0/K) = 0$ , where  $x_0$  is the spot exchange rate. The annual volatility is  $\sigma = 0.00261712\sqrt{365} = 0.05$ . We calculate  $N(-d_1)$  and  $N(-d_2)$ .

$$d_1 = \frac{(0.035 - 0.015 + 0.5(0.05^2))(0.25)}{0.05\sqrt{0.25}} = 0.2125 \quad N(-0.2125) = 0.41586$$

$$d_2 = 0.2125 - 0.05\sqrt{0.25} = 0.1875 \quad N(-0.1875) = 0.42563$$

We want an option for 120 billion yen. The value of these is \$1 billion and the strike price is \$1 billion, so the cost of the options is

$$1,000,000,000 \left( e^{-0.035(0.25)}(0.42563) - e^{-0.015(0.25)}(0.41586) \right) = \mathbf{7,618,538}$$

This answer should be rounded, since not all 7 digits are significant.

The alternative is to calculate a yen-denominated call on dollars and then to translate the price into dollars using \$1=¥120. Then the domestic rate is 0.015 and the foreign rate is 0.035, leading to the following calculation:

$$\begin{aligned}d_1 &= \frac{(0.015 - 0.035 + 0.5(0.05^2))}{0.025} = -0.1875 & N(d_1) &= 0.42563 \\d_2 &= -0.1875 - 0.025 = -0.2125 & N(d_2) &= 0.41586\end{aligned}$$

We calculate 120 billion calls, but then divide by 120 to convert to dollars, so effectively we multiply by 1 billion:

$$1,000,000,000 \left( e^{-0.035(0.25)}(0.42563) - e^{-0.015(0.25)}(0.41586) \right) = \boxed{7,618,538}$$

**8. [Subsection 10.1.1]** For a nondividend-paying stock,  $\Delta = N(d_1)$ . Since  $\Delta = N(d_1) = 0.5$ , it follows that  $d_1 = 0$  and  $d_2 = -\sigma\sqrt{t} = -0.3\sqrt{0.25} = -0.15$ . We will need to back out  $0.25r$ . We have

$$d_1 = \frac{\ln(40/41.5) + (r + 0.5(0.3^2))(0.25)}{0.3\sqrt{0.25}} = 0$$

so

$$\begin{aligned}\ln(40/41.5) + 0.25r + 0.01125 &= 0 \\0.25r &= -0.01125 - \ln(40/41.5) = 0.025564\end{aligned}$$

The price of the option is

$$40(0.5) - 41.5e^{-0.25r}N(-0.15) = 20 - 40.453N(-0.15)$$

We want to express this as one of the five choices, all of which have  $N(0.15)$ .

$$\begin{aligned}20 - 40.453N(-0.15) &= 20 - 40.453 + 40.453N(0.15) \\&= 20 - 40.453 + \frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-x^2/2} dx \\&= -20.453 + 16.138 \int_{-\infty}^{0.15} e^{-x^2/2} dx \quad \text{(D)}\end{aligned}$$

**9. [Section 12.1]** We must back out  $\sigma$ . Since  $\Delta = N(d_1) = 0.6179$ ,  $d_1 = 0.3$ .

$$\begin{aligned}\frac{(0.1 + 0.5\sigma^2)(0.25)}{\sigma(0.5)} &= 0.3 \\0.125\sigma^2 - 0.15\sigma + 0.025 &= 0 \\ \sigma &= \frac{0.15 \pm \sqrt{0.15^2 - 4(0.025)(0.125)}}{2(0.125)} = \frac{0.15 \pm 0.1}{0.25} = 0.2, 1\end{aligned}$$

The solution  $\sigma = 1$  is high and doesn't result in one of the five answer choices. On a real exam, they would tell you explicitly not to use the solution 1. Using  $\sigma = 0.2$ , the daily change resulting in no gain or loss is  $S\sigma\sqrt{h} = 50(0.2)\sqrt{1/365} = \boxed{0.5234}$ . (B)

**10. [Sections 17.1 and 23.2]**

- (i) The Black-Scholes framework includes the hypothesis that  $X(t)$  is an arithmetic Brownian motion. ✓  
 (ii) The Black-Scholes framework includes the hypothesis that volatility is constant. The variance of  $X(t+h) - X(t)$  is the square of the volatility times  $h$ , for any  $t$  and  $h$ . ✓  
 (iii) If  $dX(t) = \mu dt + \sigma dZ(t)$ , then  $(dX(t))^2 = \sigma^2 dt$  by the multiplication rules, and  $\lim_{n \rightarrow \infty} \sum_{j=1}^n dt = T$ . ✓  
 (E)

**11. [Lesson 17]**

- (i) The Black-Scholes framework includes the hypothesis that volatility is constant and its square is  $\sigma^2 h$  over an interval of length  $h$ . ✓  
 (ii) We have

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$$

$dt$  is not stochastic, while the variance of  $dZ(t)$  is  $dt$  by the multiplication rules, since  $dZ(t)$  has mean 0 and  $(dZ(t))^2 = dt$ . It follows that the variance of  $\alpha dt + \sigma dZ(t)$  is  $\sigma^2 dt$ . ✓

- (iii) If we multiply (ii) by  $S(t)$ , we have  $\text{Var}(dS(t) | S(t)) = S(t)^2 \sigma^2 dt$ , and  $dS(t) = S(t+dt) - S(t)$ .  $S(t)$  is given and therefore does not change the variance. ✓  
 (E)

**12. [Lesson 20]** Since  $d(\ln Y(t)) = \mu dt + 0.085 dZ(t)$ , it follows that  $B = 0.085$ ; the volatility of any process is equal to the volatility of the logged process. We now equate Sharpe ratios of  $X(t)$  and  $Y(t)$ .

$$\begin{aligned} \frac{0.07 - 0.04}{0.12} &= \frac{A - 0.04}{0.085} \\ 0.00255 &= 0.12(A - 0.04) \\ A &= \frac{0.00255}{0.12} + 0.04 = \boxed{0.06125} \quad (\text{B}) \end{aligned}$$

**13.** See exercise 23.3, page 454.

**14. [Section 26.3]** The Sharpe ratio does not vary with  $\alpha$ . In a Vasicek model, the Sharpe ratio is

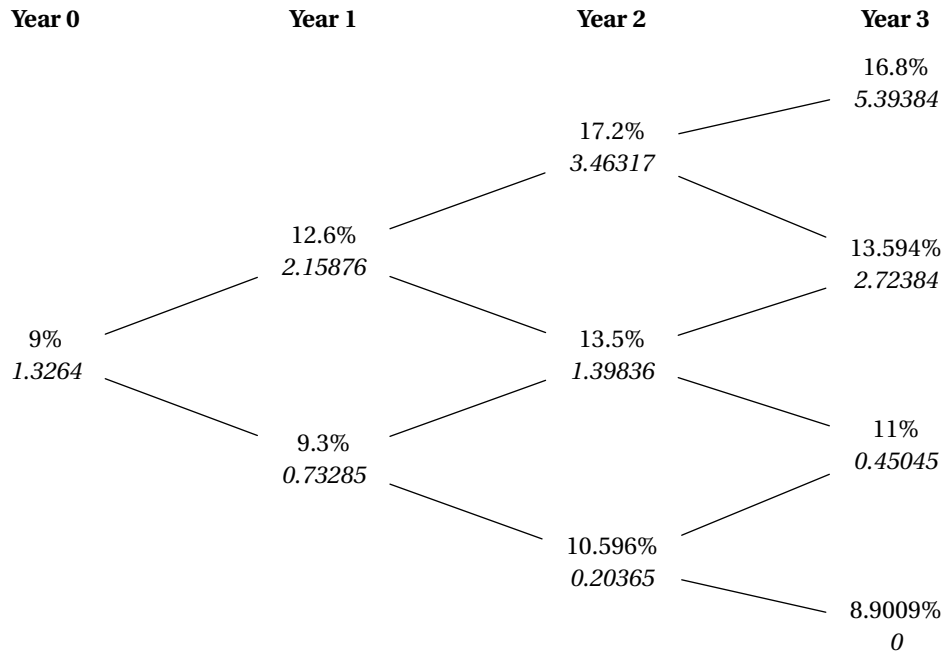
$$\phi = \frac{\alpha - r}{q} = \frac{\alpha(r, t, T) - r}{B(t, T)\sigma}$$

where  $\sigma$  is constant. This question is asking the change in  $\alpha$  as a result of changing  $r$  and  $T - t$ . We must determine the change in  $B(t, T)$ . We can express  $B(t, T)$  in terms of  $a = 0.6$  and  $T - t$  as  $\bar{a}_{\overline{T-t}|a}$ .

$$\begin{aligned} B(0, 2) &= \frac{1 - e^{-2(0.6)}}{0.6} = 1.164676 \\ B(1, 4) &= \frac{1 - e^{-3(0.6)}}{0.6} = 1.391169 \end{aligned}$$

Equating Sharpe ratios multiplied by  $\sigma$ :

$$\begin{aligned} \frac{0.04139761 - 0.04}{1.164676} &= \frac{\alpha(0.05, 1, 4) - 0.05}{1.391169} \\ \alpha(0.05, 1, 4) &= 0.05 + \frac{1.391169(0.00139761)}{1.164676} + 0.05 = \boxed{0.0516694} \end{aligned}$$



**Figure B.5:** Interest rate tree, with caplet prices, for Sample Question 15

**15. [Section 24.2]** We back out the missing interest rates on the tree, using the fact that the ratios between vertically consecutive interest rates are constant.  $R_{xxx}$  in the following refers to the effective annual interest rate at node  $xxx$ .

$$\begin{aligned}\sqrt{0.168/0.11} &= 1.235829 \\ R_{uuu} &= 0.168/1.235829 = 0.13594 \\ R_{ddd} &= 0.11/1.235829 = 0.089009 \\ 0.172/0.135 &= 1.274074 \\ R_{dd} &= 0.135/1.274074 = 0.10596\end{aligned}$$

The interest rate tree is shown in Figure B.5. The caplet prices  $C_{xxx}$  at the ending nodes are:

$$\begin{aligned}C_{uuu} &= \frac{16.8 - 10.5}{1.168} = 5.39384 \\ C_{uud} &= \frac{13.594 - 10.5}{1.13594} = 2.72384 \\ C_{udd} &= \frac{11 - 10.5}{1.11} = 0.45045\end{aligned}$$

The other caplet prices are calculated by backwards induction on the tree:

$$\begin{aligned}C_{uu} &= \frac{0.5}{1.172}(5.39384 + 2.72384) = 3.46317 \\ C_{ud} &= \frac{0.5}{1.135}(2.72384 + 0.45045) = 1.39836 \\ C_{dd} &= \frac{0.5}{1.10596}(0.45045) = 0.20365\end{aligned}$$

$$C_u = \frac{0.5}{1.126}(3.46317 + 1.39836) = 2.15876$$

$$C_d = \frac{0.5}{1.093}(1.39836 + 0.20365) = 0.73285$$

$$C = \frac{0.5}{1.09}(2.15876 + 0.73285) = \boxed{1.3264}$$



The non-year 4 nodes for which the interest rate is greater than 10.5% do not contribute to the cost of the caplet. A *caplet* is effective in only one period; here, that period is year 4. In contrast, a *cap* is effective for all periods until it expires.

The official solution's answer is slightly different since they keep less precision in the intermediate results.

The official solution also has an alternative solution which is a longcut. Namely, discount the four ending prices along each path separately. There are eight paths to the ending four nodes, each with probability 1/8. The path *ddd* can be skipped since the value on that path is 0. This longcut is not needed for this question since the value of the caplet is path-independent, but would have to be used if the payoff were path-dependent, for example if the payoff were based on an average of interest rates.

16. [Lesson 22] By formula (22.3) for the price of a prepaid forward of  $S(T)^a$ , we want

$$\exp\left((x-1)r + \frac{x(x-1)}{2}\sigma^2\right) = 1$$

or

$$(x-1)r + \frac{x(x-1)}{2}\sigma^2 = 0$$

Factoring out  $x-1$ , which generates the solution  $x=1$ , we have

$$r + \frac{x\sigma^2}{2} = 0$$

$$x = -\frac{2r}{\sigma^2}$$

For the parameters given by the question,  $x = -2(0.04)/(0.2^2) = \boxed{-2}$ . (B)

17. [Section 6.3] To make the calculation easy, the ending stock price equals the beginning stock price, so that  $\mu = 0$ , and there are only two distinct ratios of stock prices, 0.8 and 1.25, whose logs are negative each other. In fact,  $\ln 1.25 = 0.223144$ . So the unbiased sample variance over eight values is  $(8/7)(0.223144^2) = 0.056906$ . This is the monthly variance; the annual variance is  $12(0.056906) = 0.68288$ . The annual volatility is  $\sqrt{0.68288} = \boxed{0.82636}$ . (A)

18. [Subsection 14.2.3] We express the gap call as a standard call plus all-or-nothing calls.

$$C_{\text{gap}} = C(S, 100, 1) - (30 | S > 100) = C(S, 100, 1) - 30N(d_2(100))$$

Delta for this option is then the derivative with respect to  $S$ , or

$$\Delta = N(d_1(100)) - 30N'(d_2(100))d_2'(100) \quad (*)$$

We calculate  $N(d_1(100))$ ,  $N'(d_2(100))$ , and  $d_2'(100)$ .

$$\begin{aligned}d_1(100) &= \frac{0.5\sigma^2}{\sigma} = 0.5 \\N(d_1(100)) &= 0.69146 \\d_2(100) &= 0.5 - 1 = -0.5 \\N'(d_2(100)) &= \frac{e^{-(0.5^2/2)}}{\sqrt{2\pi}} = 0.352065 \\d_2(100) &= \ln S - \ln 100 + 0.5 \\d_2'(100) &= \frac{1}{S} = 0.01\end{aligned}$$

Substituting into (\*), we conclude that the number of shares needed to delta-hedge is

$$1000\Delta = 1000(0.69146 - 30(0.352065)(0.01)) = \boxed{585.84} \quad (\text{A})$$

**19. [Subsection 14.4.2]** One year from now, the call option value is as follows:

$$\begin{aligned}d_1 &= \frac{0.08 + 0.5(0.3^2)}{0.3} = 0.41667 & N(d_1) &= N(0.41667) = 0.66154 \\d_2 &= 0.41667 - 0.3 = 0.11667 & N(d_2) &= N(0.11667) = 0.54644 \\C(S, K, 0.3, 0.08, 1, 0) &= S(0.65154 - e^{-0.08}(0.54644)) = 0.15711\end{aligned}$$

Therefore, the price now is  $0.15711F_{0,1}^P(S)$ . We are given that  $F_{0,1}(S) = \$100$ , so  $F_{0,1}^P(S) = 100e^{-0.08}$  and the forward start option's value is  $100e^{-0.08}(0.15711) = \boxed{14.50}$ . (C)

**20. [Section 10.2]** Let  $\Delta_C$  and  $\Delta_P$  be the call delta and put delta respectively. From the delta of Investor B's portfolio, we conclude

$$2\Delta_C - 3\Delta_P = 3.4 \quad (*)$$

Elasticity of a portfolio is  $S\Delta_{\text{portfolio}}/C_{\text{portfolio}}$ , where  $\Delta_{\text{portfolio}}$  and  $C_{\text{portfolio}}$  are delta and the price for the entire portfolio. Let  $C$  and  $P$  be the call and put prices respectively. From the elasticity of Investor A's portfolio, we conclude

$$\begin{aligned}\frac{S(2\Delta_C + \Delta_P)}{2C + P} &= 5.0 \\ \frac{45(2\Delta_C + \Delta_P)}{2(4.45) + 1.90} &= 5.0 \\ \frac{45}{10.80}(2\Delta_C + \Delta_P) &= 5.0 \\ 2\Delta_C + \Delta_P &= 1.2 \quad (**)\end{aligned}$$

Subtracting equation (\*\*) from equation (\*), we get  $-4\Delta_P = 2.2$ , or  $\Delta_P = -0.55$ . Then the put option elasticity is  $45(-0.55)/1.90 = \boxed{-13.03}$ . (D)

**21. [Section 26.3]** In a Cox-Ingersoll-Ross model, the Sharpe ratio does not vary with  $T$  or  $t$  but is proportional to  $\sqrt{r}$ . The volatility  $\bar{\sigma}\sqrt{r}$  is proportional to  $\sqrt{r}$  as well. Also,  $q(r, t, T) = B(t, T)\sigma$ . From the definition of the Sharpe ratio and from the values we are given,

$$\phi(0.05) = \frac{\alpha(0.05, 7, 9) - r}{q(r, t, T)} = \frac{0.06 - 0.05}{B(t, T)\bar{\sigma}\sqrt{0.05}}$$

and it follows that

$$\phi(0.04) = \sqrt{\frac{0.04}{0.05}} \phi(0.05) = \sqrt{\frac{0.04}{0.05}} \frac{0.01}{B(t, T)\bar{\sigma}\sqrt{0.05}}$$

Set this equal to the definition of  $\phi(0.04)$ .

$$\frac{\alpha(0.04, 11, 13) - 0.04}{B(t, T)\bar{\sigma}\sqrt{0.04}} = \sqrt{\frac{0.04}{0.05}} \frac{0.01}{B(t, T)\bar{\sigma}\sqrt{0.05}}$$

Divide both sides by  $\sqrt{0.04}$ , and multiply both sides by  $B(t, T)\bar{\sigma}$ .

$$\begin{aligned} \frac{\alpha(0.04, 11, 13) - 0.04}{0.04} &= \frac{0.01}{0.05} = \frac{1}{5} \\ \alpha(0.04, 11, 13) &= 0.04 + 0.04/5 = \boxed{0.048} \quad (\text{C}) \end{aligned}$$

**22. [Example 26B]** To go from the true process to the risk-neutral process,  $\sigma(r, t)\phi(r, t)$  is added to the drift term. We are given that  $\sigma = 0.3$ , so we can derive  $\phi(r, t)$ :

$$\phi(r, t) = \frac{(0.15 - 0.5r(t)) - (0.09 - 0.5r(t))}{0.3} = \frac{0.06}{0.3} = 0.2$$

Since  $g(r(t), t)$  depends on the same  $dZ(t)$ , it must have the same Sharpe ratio. Let  $\mu(r, g) = m(r, g)g$ . Then

$$\begin{aligned} \frac{m(r, g) - r}{0.4} &= 0.2 \\ m(r, g) &= r + 0.08 \end{aligned}$$

and  $\mu(r, g) = \boxed{(r + 0.08)g}$ . (D)

**23. [Section 10.2, Lessons 12 and 17]** Although the question is couched in Brownian motion terminology, it is an elasticity question.

From (i) and (v), the risk premium for the stock is  $0.1 - 0.04 = 0.06$ . To delta-hedge a call option one has written, one must buy  $\Delta$  shares, which costs  $S\Delta$ . The elasticity of the call option is  $S\Delta/C$ . From (iii) and (iv), we conclude that the elasticity of the call option is  $9/6 = 1.5$ . By equation (10.7), the risk premium of the call option is  $\Omega(0.06) = 0.09$ . The rate of return on the call option is  $r$  plus the risk premium. Therefore,  $\gamma = 0.04 + 0.09 =$

**0.13**. (C)

**24.** See Example 23G, page 451.

**25. [Subsection 14.4.1]** Let  $C(t, T)$  be the price of a European call option at time  $t$  expiring at time  $T$ . The question's notation  $C(T)$  is the same as our  $C(0, T)$ .

At time 1, the chooser option is worth

$$\max(C(1, 3), P(1, 3)) = C(1, 3) + \max(0, P(1, 3) - C(1, 3)) = C(1, 3) + \max(0, K - S(1))$$

where the last equality is by put-call parity, since  $r = \delta = 0$ . Moving back to time 0, the first term becomes  $C(0, 3) = C(3)$  and the second term is a put option expiring at time 1, so the chooser option is worth

$$C(3) + P(1)$$

By put-call parity,  $P(1) = C(1) + K - S(0)$ . So we have

$$C(3) + C(1) + K - S(0) = 20$$

$$C(3) + 4 + 100 - 95 = 20$$

$$C(3) = \boxed{11} \quad (\mathbf{B})$$

**26. [Section 2.1]** A call option is worth less than the stock price, and a put option is worth less than the strike price.

A European call option is worth at least as much as implied by put-call parity against a put worth 0, or  $C \geq Se^{-\delta t} - Ke^{-rt}$ . Here,  $e^{-\delta t} = 1$  and  $Ke^{-rt} = 100e^{-0.05} = 95.12$ . Thus graph II gives bounds for the value of a European call option. An American call option on a stock with no dividends is worth the same as a European call option, so graph II gives bounds for the value of an American call option as well.

A European put option is worth at least as much as implied by put-call parity against a call worth 0, or  $P \geq Ke^{-rt} - Se^{-\delta t} = 95.12 - S$ . Thus graph IV gives bounds for the value of a European put option. An American put option can be exercised immediately so it must be worth at least its exercise value of  $K - S = 100 - S$ , so graph III gives bounds for the value of an American put option. **(D)**

**27. [Section 3.2]** We can either use a replicating portfolio or risk-neutral probabilities.

**Replicating Portfolio** Let  $\Delta_X$  and  $\Delta_Y$  be the number of shares of  $X$  and  $Y$  respectively in the replicating portfolio, and  $B$  the amount lent. Then the three outcomes imply

$$200\Delta_X + Be^{0.1} = 95 - 105 = -10 \quad (*)$$

$$50\Delta_X + Be^{0.1} = 95 \quad (**)$$

$$300\Delta_Y + Be^{0.1} = 0 \quad (***)$$

Subtracting **(\*\*)** from **(\*)**,  $150\Delta_X = -105$ , so  $\Delta_X = -0.7$  and  $Be^{0.1} = 130$ . From **(\*\*\*)**,  $300\Delta_Y + 130 = 0$ , so  $\Delta_Y = -13/30$ . Then

$$P_Y - C_X = 100(\Delta_X + \Delta_Y) + B = 100(-0.7 - 13/30) + 130e^{-0.1} = \boxed{4.2955} \quad (\mathbf{A})$$

**Risk-Neutral Probabilities** Let  $p_i^*$  be the risk-neutral probability of outcome  $i$ . For  $Y$ , we have  $300p_3^* = 100e^{0.1}$ , so  $p_3^* = e^{0.1}/3 = 0.368390$ . For  $X$ , we have

$$200p_1^* + 50p_2^* = 100e^{0.1}$$

$$200p_1^* + 50(1 - p_1^* - e^{0.1}/3) = 100e^{0.1}$$

$$150p_1^* = \frac{350e^{0.1}}{3} - 50 = 78.9366$$

$$p_1^* = \frac{78.9366}{150} = 0.526244$$

so  $p_2^* = 1 - 0.526244 - 0.368390 = 0.105366$ . Then  $P_Y - C_X = e^{-0.1}(-10p_1^* + 95p_2^*) = \boxed{4.2955}$ .

Note that the state prices in the official solution are the risk-neutral probabilities multiplied by  $e^{-0.1}$ .

**28. [Section 14.1]** The option pays 100 if  $S(1) > 10$ , otherwise 0; squaring  $S(1)$  plays no role. It is an all-or-nothing option of the form  $100 \mathbb{1}_{S(1) > 10}$ . The price of such an option is  $C = 100e^{-rt} N(d_2(10))$ . The number of shares to delta hedge is the derivative of this price, or  $100e^{-rt} N'(d_2)d_2'$ .

$$d_2 = \frac{\ln(S/10) + r + 0.5\sigma^2}{\sigma} = \frac{\ln(10/10) + 0.02 - 0.5(0.2^2)}{0.2} = 0$$

$$\begin{aligned}
 N'(0) &= \frac{1}{\sqrt{2\pi}} = 0.3989 \\
 d'_2 &= \frac{1}{S\sigma} = \frac{1}{(10)(0.2)} = 0.5 \\
 \frac{\partial C}{\partial S} &= 100e^{-0.02}(0.3989)(0.5) = \boxed{19.55} \quad (\text{A})
 \end{aligned}$$

29. [Section 24.2] First we back out  $r_{ud}$ , using the fact that the ratio of 6% to  $r_{ud}$  equals the ratio of  $r_{ud}$  to 2%:

$$r_{ud} = \sqrt{(0.06)(0.02)} = 0.03464$$

Then we compute the prices of the 3-year bond at the upper and lower Year 1 nodes,  $P_u$  and  $P_d$  respectively.

$$\begin{aligned}
 P_u &= \frac{0.5}{1.05} \left( \frac{1}{1.06} + \frac{1}{1.03464} \right) = 0.9095 \\
 P_d &= \frac{0.5}{1.03} \left( \frac{1}{1.03464} + \frac{1}{1.02} \right) = 0.9451
 \end{aligned}$$

We convert these into effective annual interest rates, which we'll call  $R_u$  and  $R_d$ .

$$\begin{aligned}
 R_u &= \left( \frac{1}{0.9095} \right)^{1/2} - 1 = 0.048573 \\
 R_d &= \left( \frac{1}{0.9451} \right)^{1/2} - 1 = 0.028635
 \end{aligned}$$

The formula for volatility is  $R_u/R_d = e^{2\sigma h}$  with  $h$  the time period. Here  $h = 1$ , and we get

$$\sigma = \frac{\ln(0.048573/0.028635)}{2} = \boxed{0.2642} \quad (\text{D})$$

30. [Section 24.2] To get 10% volatility, we need  $r_u = e^{0.2}r_d$ . To get a bond price of 88.50 for the two year bond, we need

$$(0.5)(94.34) \left( \frac{1}{1+r_d} + \frac{1}{1+r_d e^{0.2}} \right) = 88.50$$

We solve this equation for  $r_d$ . First multiply out the denominators

$$(0.5)(94.34) (2 + r_d(1 + e^{0.2})) = 88.5 (1 + r_d(1 + e^{0.2}) + r_d^2 e^{0.2})$$

Move all terms to one side to get a quadratic equation.

$$\begin{aligned}
 88.5e^{0.2}r_d^2 + (1 + e^{0.2})(88.5 - 0.5(94.34))r_d + 88.5 - 94.34 &= 0 \\
 108.09r_d^2 + 91.81r_d - 5.84 &= 0
 \end{aligned}$$

Solve

$$r_d = \frac{-91.81 + \sqrt{10954}}{2(108.09)} = \boxed{0.0594} \quad (\text{A})$$

**31. [Subsections 10.1.1 and 10.1.7]** Assume the bull spread is constructed by buying a 50-strike call and selling a 60-strike call. Then delta for the bull spread is  $N(d_1(50)) - N(d_1(60))$ . Note that the same delta would result if the bull spread were constructed by buying a 50-strike put and selling a 60-strike put, since delta for a put on a stock with no dividends is  $N(d_1) - 1$ , and  $N(d_1(50)) - 1 - (N(d_1(60)) - 1)$  is the same bull spread delta as above after canceling the 1's. To work out this question, we must compute four  $N(d_1)$ 's. Initially

$$d_1(50) = \frac{(0.05 + 0.5(0.2^2))(0.25)}{0.1} = 0.175 \quad N(d_1(50)) = N(0.175) = 0.56946$$

$$d_1(60) = \frac{\ln(5/6) + (0.05 + 0.5(0.2^2))(0.25)}{0.1} = -1.64822 \quad N(d_1(60)) = N(-1.64822) = 0.04965$$

After one month,

$$d_1(50) = \frac{(0.05 + 0.5(0.2^2))(1/6)}{0.2/\sqrt{6}} = 0.14289 \quad N(d_1(50)) = N(0.14289) = 0.55681$$

$$d_1(60) = \frac{\ln(5/6) + (0.05 + 0.5(0.2^2))(1/6)}{0.2/\sqrt{6}} = -2.09009 \quad N(d_1(60)) = N(-2.09009) = 0.01830$$

The initial delta for the bull spread is  $0.56946 - 0.04965 = 0.51981$ , and the delta after one month is  $0.55681 - 0.01830 = 0.53851$ . The difference is  $0.53851 - 0.51981 = \boxed{0.01870}$ . (B)

**32. [Lesson 21]** The fund  $W$  has volatility  $\varphi\sigma$ . Since it must have the same Sharpe ratio as  $S$ , its rate of return  $\gamma$  must satisfy

$$\frac{\gamma - r}{\varphi\sigma} = \frac{\alpha - r}{\sigma}$$

$$\gamma = r + \varphi(\alpha - r)$$

giving us the geometric Brownian motion

$$\frac{dW(t)}{W(t)} = (\varphi\alpha + (1 - \varphi)r)dt + \varphi\sigma dZ(t)$$

which isn't quite choice (A). The solution of this differential equation for  $W(t)$  is arrived at by subtracting half the volatility squared from the coefficient of  $dt$  and using that as an exponent, so

$$W(t) = W(0) \exp\left((\varphi\alpha + (1 - \varphi)r - 0.5\varphi^2\sigma^2)t + \varphi\sigma Z(t)\right)$$

which isn't quite choices (B) or (C). So we'll also have to compute  $(S(t)/S(0))^\varphi$ .

$$\left(\frac{S(t)}{S(0)}\right)^\varphi = \exp\left((\varphi\alpha - 0.5\varphi\sigma^2)t + \varphi\sigma Z(t)\right)$$

Subtract the exponent of  $(S(t)/S(0))^\varphi$  from the exponent of  $(W(t)/W(0))$ , and what remains is

$$((1 - \varphi)r - 0.5\sigma^2(\varphi^2 - \varphi))t = ((1 - \varphi)r - 0.5\sigma^2\varphi(\varphi - 1))t = (1 - \varphi)(r + 0.5\varphi\sigma^2)t$$

which is answer choice (E).

**33. [Subsection 14.4.2]** Each of the four put options is a forward start option; even the first option, purchased immediately, can be considered a forward start option with a delay of 0. The price  $P$  of each option at the time purchased is

$$\begin{aligned}d_1 &= \frac{\ln(1/0.9) + (0.08 + 0.5(0.3^2))(0.25)}{0.3\sqrt{0.25}} = 0.91074 & N(-d_1) &= N(-0.91074) = 0.18122 \\d_2 &= 0.91074 - 0.3(0.5) = 0.76074 & N(-d_2) &= N(-0.76074) = 0.22341 \\P &= S(0.9e^{-0.02}(0.22341) - 0.18122) = 0.01587S\end{aligned}$$

The sum of the cost of the four options if paid immediately is

$$0.01586(S(0) + F_{0,0.25}^P(S(0.25)) + F_{0,0.5}^P(S(0.5)) + F_{0,0.75}^P(S(0.75)))$$

However, the prepaid forward price of a nondividend paying stock is the stock price. So the amount you pay your broker is  $0.01587(45)(4) = \boxed{2.8562}$ . (C)

**34. [Section 23.2]** The answer is built into the multiplication rules: the limit of  $(dZ)^3 = (dZ)^2 \times dZ = dt \times dZ = 0$ , making the answer (A).

**35. [Lesson 23]** Having seen Example 23G, this question is a breeze. Just imitate the steps.

First, calculate  $dR(t)$ . As you know from Example 23G, the trick is to pull  $e^{-t}$  out of the integral, use the product rule, and then put most of the pieces of the result back together again as  $R(t)$ .

$$\begin{aligned}dR(t) &= -R(t)e^{-t}dt + 0.05e^{-t}dt + 0.1d\left(e^{-t}\int_0^t e^s\sqrt{R(s)}dZ(s)\right) \\&= \left(-R(t)e^{-t} + 0.05e^{-t} - 0.1e^{-t}\int_0^t e^s\sqrt{R(s)}dZ(s)\right)dt + 0.1e^{-t}e^t\sqrt{R(t)}dZ(t)\end{aligned}$$

where we evaluated  $d$  of the integral as the integrand with  $s$  replaced with  $t$ . Now, notice that the  $dt$  term is  $0.05 - R(t)$ .

$$dR(t) = (0.05 - R(t))dt + 0.1\sqrt{R(t)}dZ(t)$$

Now we move on to  $X(t)$ . Using Itô's lemma,

$$\begin{aligned}dX(t) &= \frac{\partial X(t)}{\partial R(t)}dR(t) + 0.5\left(\frac{\partial^2 X(t)}{\partial R(t)^2}\right)(dR(t))^2 + \frac{\partial X(t)}{\partial t}dt \\&= 2R(t)dR(t) + (dR(t))^2\end{aligned}$$

and plugging in  $dR(t)$ ,

$$\begin{aligned}dX(t) &= 2R(t)(0.05 - R(t))dt + 0.2R(t)\sqrt{R(t)}dZ(t) + 0.01R(t)dt \\&= (0.11R(t) - 2R(t)^2)dt + 0.2R(t)^{3/2}dZ(t) \\&= (0.11\sqrt{X(t)} - 2X(t))dt + 0.2X(t)^{3/4}dZ(t) \quad \text{(B)}\end{aligned}$$

**36. [Lesson 19 or 22]** Let  $C$  be the price of the derivative security. By the Black-Scholes equation, equation (19.1),

$$\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2} + (r - \delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} = rC$$

Let  $C = S^a$ . Then, with  $\delta = 0$ ,

$$\begin{aligned}\frac{\partial C}{\partial S} &= a(S(t))^{a-1} \\ \frac{\partial^2 C}{\partial S^2} &= a(a-1)(S(t))^{a-2} \\ \frac{\partial C}{\partial t} &= 0 \\ 0.5\sigma^2 a(a-1)(S(t))^a + ra(S(t))^a &= r(S(t))^a\end{aligned}$$

Divide out  $(S(t))^a$ .

$$0.5a(a-1)\sigma^2 + ra - r = 0 \quad (*)$$

$a = 1$  is one solution. Dividing it out, we have  $0.5a\sigma^2 + r = 0$ , or  $a = -2r/\sigma^2$ . In our case,  $a = -k/\sigma^2$  and  $r = 0.04$ , so  $a = -0.08/\sigma^2$  and  $k = \boxed{0.08}$ . (E)

Another way to get the same answer is to realize that this security is  $S^a$  for  $a = -k/\sigma^2$ . What would the prepaid forward price of this security be? Well, whatever it is, it would not vary with  $t$ , since regardless of the time of maturity of the forward, you would receive a security with the same value. But in formula (22.3), the prepaid forward on  $S^a$  has a value varying with  $T$ , namely  $S(0)^a e^{ut}$ , where  $u$  is an expression involving  $a$ ,  $r$ , and  $\delta$ . The only way to reconcile with that formula is to have  $u = 0$ . So we get the equation

$$u = a(r - \delta) + 0.5a(a-1)\sigma^2 - r = 0$$

which with  $\delta = 0$  is the same equation as (\*).

**37. [Section 13.1]** The trick is to express the  $S(t)$ 's in terms of  $Q(t)$ 's where  $Q(t) = S(t)/S(t-1)$ . So

$$\begin{aligned}G &= \left( (S(0)Q(1))(S(0)Q(1)Q(2))(S(0)Q(1)Q(2)Q(3)) \right)^{1/3} \\ &= S(0)Q(1)Q(2)^{2/3}Q(3)^{1/3}\end{aligned}$$

The  $Q(t)$ 's are independent lognormal random variables with  $m = 0.03$  and  $v = 0.2$ . Logging them yields normal random variables with  $\mu = 0.03 - 0.5(0.2^2) = -0.01$  and  $\sigma = 0.2$ . Then

$$\begin{aligned}\ln G &= \ln S(0) + \ln Q(1) + \frac{2}{3} \ln Q(2) + \frac{1}{3} \ln Q(3) \\ \text{Var}(\ln G) &= \text{Var}(\ln Q(1)) + \frac{4}{9} \text{Var}(\ln Q(2)) + \frac{1}{9} \text{Var}(\ln Q(3)) \\ &= 0.04 \left( 1 + \frac{4}{9} + \frac{1}{9} \right) = \boxed{0.06222} \quad (\text{D})\end{aligned}$$

**38. [Section 26.5]** Delta is the first derivative of  $P$  with respect to  $r$  and gamma is the second derivative, and we plug in  $r = 0.05$ :

$$\begin{aligned}P &= Ae^{-Br} \\ \Delta &= -BP \\ \Gamma &= B^2P\end{aligned}$$

The delta-gamma approximation is (equation (26.21) without  $\theta$ )

$$\begin{aligned}P(t, T, r) &= P(t, T, r_0) + \Delta(r - r_0) + 0.5\Gamma(r - r_0)^2 \\ \frac{P_{\text{Est}}(0, 3, 0.03)}{P(0, 3, 0.05)} &= 1 + (-0.02)(-B) + 0.5(0.02^2)(B^2) \\ &= 1 + 0.02(2) + 0.5(0.02^2)(2^2) = \boxed{1.0408} \quad (\text{B})\end{aligned}$$

**39. [Lesson 3]** This is a sophisticated question combining put-call parity, binomial trees, and interest rate binomial trees.

From put-call parity,  $P(108) - C(108) = F^P(108) - S$ . The prepaid forward price of 108 is the probability-weighted discounted value of 108 paid two years from now. We must consider the two possibilities of  $u$  and  $d$ . To determine the probabilities, we use the stock prices. Since the stock pays no dividends,

$$p^* = \frac{1.05S - S_d}{S_u - S_d} = \frac{105 - 95}{110 - 95} = \frac{2}{3}$$

With  $2/3$  probability the two-year discounting rate is  $1/(1.05)(1.06)$  and with  $1/3$  probability the two-year discounting rate is  $1/(1.05)(1.04)$ . Therefore

$$F^P(108) = 108 \left( \frac{2}{3} \left( \frac{1}{(1.05)(1.06)} \right) + \frac{1}{3} \left( \frac{1}{(1.05)(1.04)} \right) \right) = 97.6571$$

$$P(108) - C(108) = 97.6571 - 100 = \boxed{-2.34} \quad (\mathbf{B})$$

**40. [Subsection 1.1.3 and Section 11.1]** The profit diagrams are interesting, but for answering the question, it suffices to look at the hatched light blue expiration graph. Portfolio I's expiration looks like a collar. Portfolio II's expiration has the signature look of a straddle, which gains with volatility. Portfolio III's expiration looks like a strangle, and Portfolio IV's expiration has the increasing Z shape of a bull spread. **(D)** See Subsection 1.1.3 for diagrams of the payoffs at expiration of these option strategies.

**41. [Subsection 10.1.1 and Section 13.3]** I did this question using the method in the official solution's remark rather than the main method, although they're both equally difficult.

Let  $V$  be the value of the contingent claim at time 0. The payoff is  $\min(S(1), 42) = S(1) - \max(0, S(1) - 42)$ . Discounting to time-0, this becomes  $V = Se^{-\delta} - C(S, 42, 1) = Se^{-0.03} - C(S, 42, 1)$ . Another way to derive this is to realize that this contingent claim is a bull spread with strikes 0 and 42, and a call with strike price 0 is the stock itself.

Delta for this contingent claim is the derivative of  $V$  with respect to  $S$ , or  $\Delta_{\text{contingent claim}} = e^{-\delta} - e^{-\delta}N(d_1)$ , since we know that  $\Delta$  for the call is  $e^{-\delta}N(d_1)$  (equation (10.2)). Now let's compute  $C(S, 42, 1)$  and  $N(d_1)$ .

$$d_1 = \frac{\ln(45/42) + 0.07 - 0.03 + 0.5(0.25^2)}{0.25} = 0.56097 \quad N(d_1) = N(0.56097) = 0.71259$$

$$d_2 = 0.56097 - 0.25 = 0.31097 \quad N(d_2) = N(0.31097) = 0.62209$$

$$C(S, 42, 1) = 45e^{-0.03}(0.71259) - 42e^{-0.07}(0.62209) = 6.75746$$

The elasticity is

$$\Omega = \frac{S\Delta_{\text{contingent claim}}}{V} = \frac{45(e^{-0.03} - e^{-0.03}(0.71259))}{45e^{-0.03} - 6.75746} = \boxed{0.34003} \quad (\mathbf{C})$$

**42. [Section 13.2]** For  $H = \infty$ , an up-and-out call is a standard European call.

The special option of this question is equivalent to two up-and-in calls with barrier 70 minus one up-and-in call with barrier 80. An up-and-in call is equal to a standard call minus an up-and-out call. If we let  $C(H)$  be the value of an up-and-out call with barrier  $H$ , then the value  $V$  of the special option is

$$V = 2(C(\infty) - C(70)) - (C(\infty) - C(80)) = C(\infty) - C(70) + C(80) = 4.0861 - 2(0.1294) + 0.7583 = \boxed{4.5856} \quad (\mathbf{D})$$

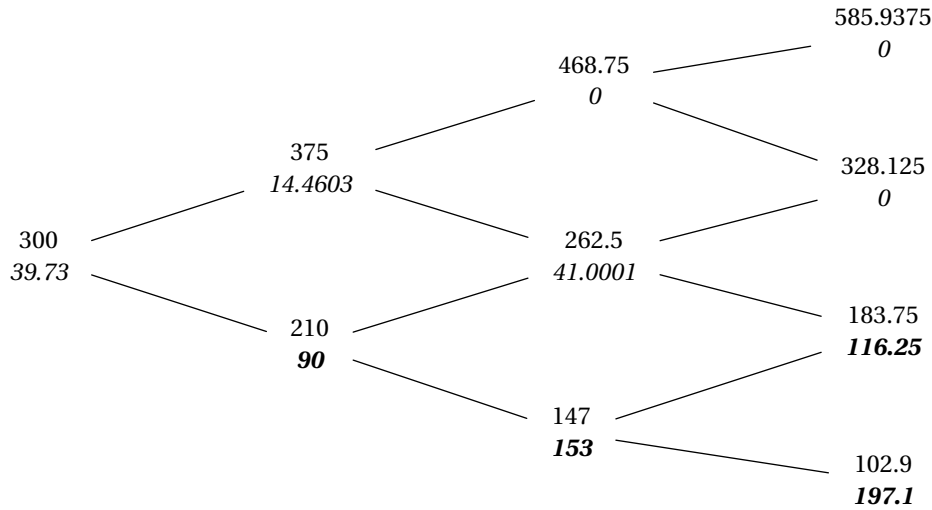


Figure B.6: Binomial tree for Sample Question 44

43. [Lesson 22] The fastest way to do this is to use the formula for the Itô process of  $S^a$ , equation (22.4), with  $a = -1$ . Here,  $\delta = r_{\epsilon}$ . So we have

$$\begin{aligned} \frac{dy(t)}{y(t)} &= ((-1)(r - r_{\epsilon}) + 0.5(-1)(-2)\sigma^2)dt + (-1)\sigma dZ(t) \\ &= (r_{\epsilon} - r + \sigma^2)dt - \sigma dZ(t) \quad \text{(E)} \end{aligned}$$

44. [Section 4.1] In this tree,  $u = 1.25$  and  $d = 0.7$  at all nodes, although that is not needed to solve the question. The risk-neutral probability at all nodes is

$$p^* = \frac{300e^{0.1-0.065} - 210}{375 - 210} = 0.610218$$

At the ending nodes, the put payoffs are, from top to bottom, 0, 0, 116.25, 197.1. The tree is shown in Figure B.6.

Pulling back to the third column of nodes:

$$\begin{aligned} P_{uu} &= 0 \\ P_{ud} &= e^{-0.1}(0.389782)(116.25) = 41.0001 \\ P_{dd}^{\text{tentative}} &= e^{-0.1}((0.610218)(116.25) + (0.389782)(197.10)) = 133.7023 \end{aligned}$$

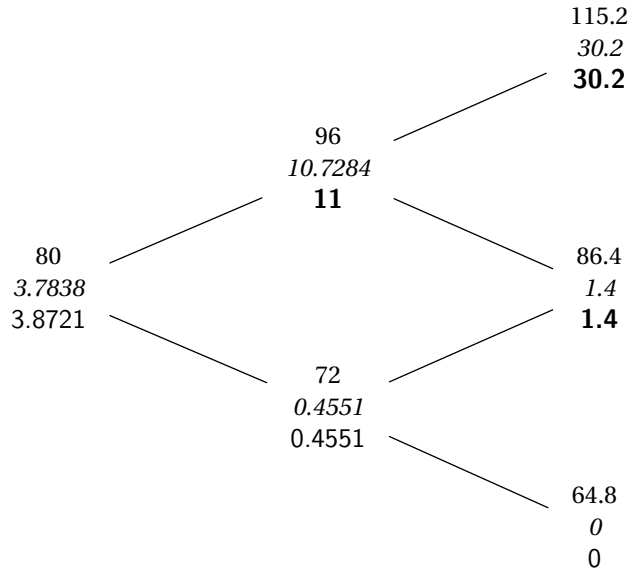
At the  $dd$  node, the exercise value  $300 - 147 = 153$  is greater than the calculated value, so early exercise is optimal.

Pulling back to the second column of nodes:

$$\begin{aligned} P_u &= e^{-0.1}(0.389782)(41.0001) = 14.4603 \\ P_d^{\text{tentative}} &= e^{-0.1}((0.610218)(41.0001) + (0.389782)(153)) = 76.5997 \end{aligned}$$

At the  $d$  node, the exercise value  $300 - 210 = 90$  is greater than the calculated value, so early exercise is optimal.

Initially,  $P = e^{-0.1}((0.610218)(14.4603) + (0.389782)(90)) = \boxed{39.73}$ . (D)



**Figure B.7:** Binomial tree for Sample Question 46. At each node, the first number is the future price, the second number is the European option price, and the third number is the American option price. The American option price is boldface if it is optimal to exercise early.

**45. [Section 12.3]** We calculate gamma by calculating delta at the two Year 1 nodes and then dividing the change in delta by the change in the stock price. The formula for delta is equation (3.1). Although the letter  $C$  is used in that formula, it applies equally well to puts. The formula for gamma is equation (12.5).

$$\begin{aligned}\Delta_u &= e^{-0.065} \frac{0 - 41.0001}{468.75 - 262.5} = -0.186278 \\ \Delta_d &= e^{-0.065} \frac{41.0001 - 153}{262.5 - 147} = -0.908670 \\ \Gamma &= \frac{(-0.186278) - (-0.908670)}{375 - 210} = \boxed{0.004378} \quad (C)\end{aligned}$$

**46. [Section 4.4]** For a futures contract, the dividend rate is set equal to the risk-free rate in the formulas, so  $p^* = (1 - d)/(u - d)$ . Therefore,  $u$  and  $d$  are computed as follows:

$$\begin{aligned}\frac{1}{3} &= \frac{1 - d}{u - d} = \frac{1 - d}{d/3} \\ 3 - 3d &= \frac{d}{3} \\ 9 &= 10d \\ d &= 0.9 \\ u &= 1.2\end{aligned}$$

The binomial tree is shown in Figure B.7.

The calculation of the second column is:

$$C_u^{\text{European}} = e^{-0.025} \left( \frac{1}{3}(30.2) + \frac{2}{3}(1.4) \right) = 10.7284$$

$$C_d = e^{-0.025} \left( \frac{1}{3}(1.4) \right) = 0.4551$$

At the  $u$  node, the exercise value is  $96 - 85 = 11$ , so it is optimal to exercise. This is the only node where early exercise is optimal. We could therefore answer the question without calculating the initial option values by discounting the difference of the top node:  $e^{-0.025} \left( \frac{1}{3}(11 - 10.7284) \right) = \boxed{0.0883}$ . (E) However, the tree shows all of the option values.

47. [Section 12.1] Since they give you the put option price, you can be pretty sure that put-call parity will be used.

The initial cash flow to the female investor for selling 100 call options and buying 79.4 (100 $\Delta$ ) shares of stock is

$$100(8.88) - 79.4(40) = -2288$$

The final cash flow at the close of the position, which means buying 100 call options and selling 79.4 shares of stock is

$$-100(14.42) + 79.4(50) = 2528$$

To calculate interest, we use put-call parity. Let  $t_1$  be the initial time,  $t_2$  the time of closing the position, and  $T$  the expiry time for the options. Then

$$\text{Initially:} \quad 8.88 - 1.63 = 7.25 = 40 - Ke^{-r(T-t_1)} \quad \text{or} \quad Ke^{-r(T-t_1)} = 32.75$$

$$\text{At closing time:} \quad 14.42 - 0.26 = 14.16 = 50 - Ke^{-r(T-t_2)} \quad \text{or} \quad Ke^{-r(T-t_2)} = 35.84$$

Dividing the first equation into the second yields  $e^{r(t_2-t_1)} = 35.84/32.75 = 1.09435$ . The investor's profit is  $2528 - 2288(1.09435) = \boxed{24.12}$ . (B)

Student reports indicated that this question appeared on the Fall 2007 exam.

48. [Lesson 20] There is no need to calculate  $k$ , although you could calculate  $k$  by equating Sharpe ratios. If you are interested in a solution using Sharpe ratios, see the first sample solution that comes with the sample questions.

A risk-free portfolio must have a risk premium of 0. The risk premium is  $0.06 - 0.04 = 0.02$  for  $S_1$  and  $0.03 - 0.04 = -0.01$  for  $S_2$ . Let  $x$  be the number of shares of Stock 2 to purchase.

$$0.02(100) - 0.01(50x) = 0$$

$$2 - 0.5x = 0$$

$$x = \boxed{4} \quad \text{(E)}$$

Student reports indicated that this question appeared on the Fall 2008 exam.

49. [Lesson 3] The binomial tree will have

$$u = e^{0.04(0.25) + 0.3\sqrt{0.25}} = e^{0.16}$$

$$d = e^{0.01 - 0.15} = e^{-0.14}$$

$$1 - p^* = \frac{1}{1 + e^{-\sigma\sqrt{h}}} = \frac{1}{1 + e^{-0.15}} \quad \text{(Equation (3.7))}$$

Note that  $100u = 100e^{0.16} = 117.35 < 118$ . When the option pays off at both nodes, it is optimal to exercise early since the option no longer has any risk. Let's assume the  $K < 117.35$  so that the option does not pay at the upper node. For optimal early exercise, we need the current payoff to be worth more than the discounted payoff at the lower node, or

$$K - 100 \geq e^{-0.01} (1 - p^*) (K - 100e^{-0.14}) = e^{-0.01} \left( \frac{K - 100e^{-0.14}}{1 + e^{-0.15}} \right)$$

$$\begin{aligned} &\geq \frac{Ke^{-0.01} - 100e^{-0.15}}{1 + e^{-0.15}} \\ (K - 100)(1 + e^{-0.15}) &\geq Ke^{-0.01} - 100e^{-0.15} \\ K(1 + e^{-0.15} - e^{-0.01}) &\geq 100(1 + e^{-0.15} - e^{-0.15}) = 100 \\ K &\geq \frac{100}{1 + e^{-0.15} - e^{-0.01}} = \frac{100}{0.870658} = 114.86 \end{aligned}$$

so the answer is **115**. (B)

50. [Section 7.2] The parameters of the lognormal ratio  $S_{1/2}/S_0$  are

$$\begin{aligned} m &= 0.5(0.15 - 0.5(0.35^2)) = 0.044375 \\ v &= 0.35\sqrt{0.5} = 0.247487 \end{aligned}$$

The upper bound of the 90% confidence interval for the normal distribution is  $0.044375 + 1.64485(0.247487) = 0.451454$ . The upper bound of the 90% confidence interval for the stock price after six months is  $0.25e^{0.451454} =$  **0.39265**. (A)

51. [Section 8.1] A statistical calculator may help for this question.

We calculate  $\ln(S_t/S_{t-1})$  for times  $t = 2$  through  $t = 7$ .

Month	2	3	4	5	6	7
$S_t/S_{t-1}$	0.03637	-0.15415	0.13613	0.08701	-0.03390	0.06669

$\ln(S_t/S_{t-1})$  is assumed to follow a normal distribution. We estimate the mean of the normal as the sample mean  $\mu = \bar{x} = 0.02303$  and the standard deviation as the sample standard deviation (dividing by  $n - 1 = 5$ )  $\sigma = 0.10354$ . These are per month. The monthly return is therefore  $\alpha/12 = \mu + 0.5\sigma^2 = 0.02303 + 0.5(0.10354^2) = 0.02839$ . The annual return is  $\alpha = 12(0.02839) =$  **0.3406**. (E) You can also reverse the order of these operations: first annualize  $\mu$  (multiply by 12) and  $\sigma$  (multiply by  $\sqrt{12}$ ) and then calculate  $\alpha = \mu + 0.5\sigma^2$ . It is strange that the answer is so far out of the ranges; they were thinking you'd forget to add  $0.5\sigma^2$ .

52. [Section 15.3] For 2 years, the mean and variance of the lognormal are

$$\begin{aligned} m &= 2(0.15 - 0.5(0.3^2)) = 0.21 \\ v &= 0.30\sqrt{2} = 0.4243 \end{aligned}$$

Using the inversion method, the standard normal random numbers are  $N^{-1}(0.9830) = 2.12$ ,  $N^{-1}(0.0384) = -1.77$ , and  $N^{-1}(0.7794) = 0.77$ . The resulting ratios of  $S_1/S_0$  are

$n_i$	$m + n_i v$	$e^{m+n_i v}$
2.12	1.1094	3.0327
-1.77	-0.5410	0.5822
0.77	0.5367	1.7103

The average, multiplied by  $S_0 = 50$ , is  $50(3.0327 + 0.5822 + 1.7103)/3 =$  **88.75**. (C)

53. [Section 14.1] We can express the gap option as a standard European call with strike price 40 minus a cash-or-nothing option paying 5 if the stock price is above 40. Now the two properties of gamma to use are:

(i) Gamma is a linear function of the options, since it is a second derivative with respect to the stock price,

so a linear combination of the two options will have a gamma that is the linear combination of the gammas.

- (ii) Gamma for a put equals gamma for the corresponding call, the one with the same strike price and expiry.

Thus gamma for a call plus gamma for a cash-or-nothing of 5 equals gamma for the gap call.

$$\begin{aligned} 0.08 + \text{gamma cash-or-nothing} &= 0.07 \\ \text{gamma cash-or-nothing} &= -0.01 \end{aligned}$$

For cash-or-nothing of 1000, gamma is  $(1000/5)(-0.01) = \boxed{-2}$ . (B)

Alternatively, write down all the information using our all-or-nothing notation:

$$\begin{aligned} \text{(ii) \& (iv):} \quad 40 | S < 40 - S | S < 40 \quad \Gamma &= 0.07 \\ \text{(iii) \& (v):} \quad S | S > 40 - 45 | S > 40 \quad \Gamma &= 0.08 \end{aligned} \quad (*)$$

and we need gamma for  $1000 | S > 40$ . Since we need the condition  $S > 40$ , we will replace the first equation above with a call equation:

$$S | S > 40 - 40 | S > 40 \quad \Gamma = 0.07 \quad (**)$$

Subtracting (\*) from (\*\*), we get  $5 | S > 40$  has  $\Gamma = -0.01$ , so  $1000 | S > 40$  has  $\Gamma = 200(-0.01) = \boxed{-2}$ . (B)

**54. [Sections 1.2, 13.3, and 14.3]** We want the value of an option with payoff  $\max(17 - \min(2S_1(1), S_2(1)), 0)$ , since the option holder will sell the cheapest stock for 17.

Let  $M$  be the value at time 1 of  $\min(2S_1(1), S_2(1))$ . Then the option given in (vi)—call its value  $V_1$ —pays  $\max(M - 17, 0)$  and the option we want to value—call its value  $V_2$ —pays  $\max(17 - M, 0)$ . Since one option pays exactly when the other one doesn't, the difference, or  $V_1 - V_2$ , is equal to the present value of  $M - 17$ . The present value of 17 is  $17e^{-0.05}$ . The present value of  $M$  is an option paying the minimum of  $2S_1$  and  $S_2$ , which is  $S_2 - C(S_2, 2S_1)$ , where  $C(S_2, 2S_1)$  is an exchange option which allows one to buy  $S_2$  in exchange for  $2S_1$ . Let's value this exchange option. The volatility of the difference in returns between the two stocks is the square root of

$$\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = 0.18^2 + 0.25^2 - 2(-0.40)(0.18)(0.25) = 0.13090$$

and  $\sqrt{0.13090} = 0.36180$ . By Black-Scholes, since both dividend rates are 0,

$$\begin{aligned} d_1 &= \frac{\ln(20/20) + 0.5\sigma^2}{\sigma} = 0.5(0.36180) = 0.18090 \\ d_2 &= 0.18090 - 0.36180 = -0.18090 \\ N(d_1) &= N(0.18090) = 0.57178 \\ N(d_2) &= N(-0.18080) = 0.42822 \\ C(S_2, 2S_1) &= 20(0.57178 - 0.42822) = 2.871206 \end{aligned}$$

so the present value of  $M$  is  $20 - 2.87120 = 17.12880$ . Then

$$\begin{aligned} V_1 - V_2 &= 17.12880 - 17e^{-0.05} \\ 1.632 - V_2 &= 17.12880 - 17e^{-0.05} = 0.95790 \\ V_2 &= 1.632 - 0.95790 = \boxed{0.67410} \quad (\text{A}) \end{aligned}$$

55. [Section 9.3] We must back out  $\sigma$ . For an at-the-money future,  $d_1 = -d_2$ . Therefore,

$$\begin{aligned} 1.625 &= 20e^{-0.75(0.1)}(N(-d_2) - N(-d_1)) \\ &= 20e^{-0.075}(N(d_1) - (1 - N(d_1))) = 20e^{-0.075}(2N(d_1) - 1) \\ 2N(d_1) &= \frac{1.625e^{0.075}}{20} + 1 = 1.087578 \\ N(d_1) &= 0.54379 \\ d_1 &= 0.10999 = 0.5\sigma\sqrt{3/4} \\ \sigma &= \frac{0.21998}{\sqrt{3/4}} = 0.254011 \end{aligned}$$

Now we can value the option three months later using the Black formula. With six months to expiry:

$$\begin{aligned} d_1 &= \frac{\ln(17.7/20) + 0.5(0.254011^2)(0.5)}{0.254011\sqrt{0.5}} = -0.59037 \\ d_2 &= -0.59037 - 0.254011\sqrt{0.5} = -0.76998 \\ N(-d_1) &= N(0.59037) = 0.72253 \\ N(-d_2) &= N(0.76998) = 0.77934 \\ P(F, 20, 0.5) &= 20e^{-0.5(0.1)}(0.77934) - 17.7e^{-0.5(0.1)}(0.72253) = \boxed{2.66156} \quad (\text{D}) \end{aligned}$$

56. [Sections 7.1 and 13.1] Nothing about this question requires knowledge of average strike options. You just have to evaluate the variance of  $A(2)$ .

The stock price is 5 times a lognormal random variable with parameters  $m = (0.05 - 0.5(0.2^2))t = 0.03t$  and  $v = 0.2\sqrt{t}$ . Therefore,

$$A(2) = \frac{1}{2}(S(1) + S(2)) = \frac{S(0)}{2} \left( \frac{S(1)}{S(0)} + \frac{S(2)}{S(0)} \right)$$

While  $S(1)/S(0)$  and  $S(2)/S(0)$  are not independent because the periods  $(0, 1)$  and  $(0, 2)$  overlap, the variables  $X = S(1)/S(0)$  and  $Y = S(2)/S(1)$  are independent, so we set  $S(2)/S(0) = XY$ , and

$$A(2) = \frac{5}{2}(X + XY)$$

where  $X$  and  $Y$  are independent lognormal random variables with parameters  $m = 0.03$  and  $v = 0.2$ .

We will use the two alternative methods as the official solution goes through: (1) calculating first and second moments, and (2) calculating variance of a sum directly.

First let's calculate the variance as the second moment minus the first moment squared. We can hold off the  $5/2$  until the end.

$$\begin{aligned} \mathbf{E}[X + XY] &= \mathbf{E}[X] + \mathbf{E}[XY] = e^{0.05} + e^{0.10} \\ \mathbf{E}[(X + XY)^2] &= \mathbf{E}[X^2] + \mathbf{E}[(XY)^2] + 2\mathbf{E}[X^2Y] \end{aligned}$$

$X$  and  $Y$  are independent, so the expectations may be factored. Also,  $\mathbf{E}[Z^2] = e^{2m+2v^2}$  for a lognormal  $Z$ . So

$$\begin{aligned} \mathbf{E}[X^2] &= e^{2(0.03)+2(0.2^2)} = e^{0.14} \\ \mathbf{E}[(XY)^2] &= \mathbf{E}[X^2]\mathbf{E}[Y^2] = e^{0.14}e^{0.14} = e^{0.28} \\ \mathbf{E}[X^2Y] &= \mathbf{E}[X^2]\mathbf{E}[Y] = e^{0.14}e^{0.05} = e^{0.19} \end{aligned} \quad (*)$$

Therefore the variance is

$$\begin{aligned}\text{Var}(X + XY) &= e^{0.14} + e^{0.28} + 2e^{0.19} - (e^{0.05} + e^{0.10})^2 \\ &= 4.891903 - 2.156442^2 = 0.241661\end{aligned}$$

Multiplying by  $(5/2)^2$ , the answer is  $(25/4)(0.241661) = \boxed{1.51}$ . (A)

The other method is to calculate variance directly using the variance of a sum formula:

$$\text{Var}(X + XY) = \text{Var}(X) + \text{Var}(XY) + 2\text{Cov}(X, XY)$$

Since  $X$  and  $Y$  are independent, their product is a lognormal with  $m = 0.06$  and  $v^2 = 2(0.2^2) = 0.08$ , so

$$\begin{aligned}\text{Var}(X) &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 = e^{0.14} - e^{0.1} \\ \text{Var}(XY) &= e^{0.28} - e^{0.2}\end{aligned}$$

For the covariance, evaluate it as

$$\begin{aligned}\text{Cov}(X, XY) &= \mathbf{E}[X^2 Y] - \mathbf{E}[X] \mathbf{E}[XY] \\ \mathbf{E}[X^2 Y] &= e^{0.19} \quad \text{This was evaluated above, see (*).} \\ \mathbf{E}[X] &= e^{0.05} \\ \mathbf{E}[XY] &= e^{0.1} \\ \text{Cov}(X, XY) &= e^{0.19} - e^{0.15}\end{aligned}$$

So the variance of  $X + XY$  is

$$\text{Var}(X + XY) = e^{0.14} - e^{0.1} + e^{0.28} - e^{0.2} + 2(e^{0.19} - e^{0.15}) = 0.241661$$

and multiplying by  $(5/2)^2$ , the answer again is  $(25/4)(0.241661) = \boxed{1.51}$ .

**57. [Section 15.5]**  $U_1$  and  $U_5$  get mapped to the first stratum or  $[0, 0.25)$  and  $U_4$  and  $U_8$  get mapped to the last stratum, or  $[0.75, 1)$ , so the lowest number must come from  $U_1$  or  $U_5$  and the highest one from  $U_4$  or  $U_8$ . Of  $U_1$  and  $U_5$ ,  $U_5$  is lower. Using inversion,  $N^{-1}(0.3172/4) = N(0.0793) = -1.40980$ . Of  $U_4$  and  $U_8$ ,  $U_4$  is higher. Using inversion,  $N^{-1}(0.75 + 0.4482/4) = N^{-1}(0.86205) = 1.08958$ . The difference is  $1.08958 - (-1.40980) = \boxed{2.49938}$ . (E)

**58. [Section 15.4]**  $C(40)$  is not random, so the variance of the expression for  $C^*(42)$  is

$$\hat{\text{Var}}(\hat{C}(42)) + \beta^2 \text{Var}(\hat{C}(40)) + 2\beta \text{Cov}(\hat{C}(40), \hat{C}(42))$$

The minimum of a quadratic is  $\beta = -b/2a$ , or  $\text{Cov}(\hat{C}(40), \hat{C}(42)) / \text{Var}(\hat{C}(40))$ . This is estimated with the sample covariance and variance. The simulated payoffs are

Price	Strike 40	Strike 42	Strike 40 squared	Product
33.29	0	0	0	0
37.30	0	0	0	0
40.35	0.35	0	0.1225	0
43.65	3.65	1.65	13.3225	6.0225
48.90	8.90	6.90	79.21	61.41
Sum	12.90	8.55	92.6550	67.4325

We don't have to discount by  $e^{-0.25(0.08)}$  because the discount factor appears in both the numerator and denominator and cancels. Using standard formulas for covariance and variance, each multiplied by 5,

$$\beta = \frac{\text{Cov}(\hat{C}(40), \hat{C}(42))}{\text{Var}(\hat{C}(40))} = \frac{67.4325 - (12.90)(8.55)/5}{92.6550 - 12.90^2/5} = \boxed{0.764211} \quad (\text{B})$$

The official answer points out that you can perform the regression on a statistical calculator without going through the calculations here.

59. [Section 15.4] For each  $\hat{C}(K)$ , we discount the average simulated value.

$$\hat{C}(42) = e^{-0.08(0.25)}(8.55/5) = 1.6761$$

$$\hat{C}(40) = e^{-0.08(0.25)}(12.90/5) = 2.5289$$

$$C^*(42) = 1.6761 + 0.764211(2.7847 - 2.5289) = \boxed{1.8716} \quad (\text{B})$$

60. [Section 26.2] You're given that the limit of the price of a bond at infinity is  $e^{-0.1T}$ , so the yield of an infinitely lived bond is 0.1. The formula for this yield (see the last line of Table 26.3) is

$$\bar{r} = \frac{2ab}{a - \bar{\phi} + \gamma} = \frac{2ab}{(a - \bar{\phi}) + \sqrt{(a - \bar{\phi})^2 + 2\bar{\sigma}^2}}$$

and in our case, since

$$dr(t) = 0.1(0.11 - r(t))dt + 0.08\sqrt{r(t)}dZ(t)$$

we have  $a = 0.1$ ,  $b = 0.11$ ,  $\bar{\sigma} = 0.08$ . It is easier to let the unknown be  $x = a - \bar{\phi}$  rather than  $\bar{\phi}$ , so that we don't have to expand  $(a - \bar{\phi})^2$ . Then

$$\begin{aligned} 0.1 &= \frac{2(0.1)(0.11)}{x + \sqrt{x^2 + 2(0.08^2)}} = \frac{0.022}{x + \sqrt{x^2 + 0.0128}} \\ 1 &= \frac{0.22}{x + \sqrt{x^2 + 0.0128}} \\ x + \sqrt{x^2 + 0.0128} &= 0.22 \\ x^2 + 0.0128 &= (0.22 - x)^2 = x^2 - 0.44x + 0.0484 \\ 0.44x &= 0.0484 - 0.0128 = 0.0356 \\ x &= 0.080909 \end{aligned}$$

Then  $\bar{\phi} = 0.1 - x = 0.019091$ . Since  $\phi(r, t) = \bar{\phi}\sqrt{r}/\bar{\sigma}$ ,  $c = \bar{\phi}/\bar{\sigma} = 0.019091/0.08 = \boxed{0.2386}$ . (E) The official solution points out that you don't have to solve for  $\bar{\phi}$ ; you can just plug in the five answer choices (after transforming them to  $\bar{\phi}$  by multiplying by 0.08).

61. [Section 21.1] The relationship between the risk-neutral process and the true process is  $\tilde{Z}(t) = Z(t) + \phi t$ , where  $\phi$  is the Sharpe ratio, as discussed in Section 21.1. Then

$$\mathbf{E}^*[Z(t)] = \mathbf{E}^*[\tilde{Z}(t)] - \phi t$$

where  $\mathbf{E}^*$  indicates expectation under the risk-neutral measure. Under the risk-neutral measure,  $\tilde{Z}(t)$  is a Brownian motion, so  $\mathbf{E}^*[\tilde{Z}(t)] = 0$ . We are given  $\mathbf{E}^*[Z(0.5)] = -0.03$ , so  $-0.5\phi = -0.03$  and  $\phi = 0.06$ . Now,  $\phi = (\alpha - r)/\sigma$ , and we see from the stock-price process that the rate of stock price appreciation is  $\alpha - \delta = 0.05$  and  $\sigma = 0.25$ . We are also given  $\delta = 0.01$ , so  $\alpha = 0.06$  and  $\phi = 0.06 = (0.06 - r)/0.25$ , so  $r = \boxed{0.045}$ . (D)

**62. [Lesson 22]** We'll use the obvious generalization of equation (22.2), replacing the subscript 0 with  $t$  and the exponent  $T$  with  $T - t$ . Let  $a = 2$ . Then

$$F_{t,T}(S^2) = S^2 e^{(2(r-\delta)+\sigma^2)(T-t)}$$

Here,  $r - \delta = \mu$ , so the exponent is  $(2\mu + \sigma^2)(T - t)$ , which we're given is  $0.18(T - t)$ . Since  $\sigma = 0.4$ , it follows that  $\mu = \boxed{0.01}$ . (A)

**63. [Section 23.2]** The quadratic variation is defined by

$$V_T^2(U) = \lim_{n \rightarrow \infty} \sum_{i=1}^{[nT]} \left( U(ih) - U(i(h-1)) \right)^2$$

where  $h = 1/n$ .

For (i) and (iii), we can use the shortcut of  $d$ 'ing the process and squaring it and integrating from 0 to 2.4.

For (i),  $dW(t) = 2t dt$ , and  $(2t dt)^2 = 0$  since  $(dt)^2 = 0$ .

For (iii),  $dY(t) = 2dt + 0.9dZ(t)$ , and when we square this, since  $(dZ(t))^2 = dt$ , the square of that is  $0.81dt$ . Integrating that from 0 to 2.4, we get  $2.4(0.81) = 1.944$ .

(ii) is not continuous, but it is easy to see that the variations only occur at 1 and 2, regardless of the value of  $n$  in the sum defining the quadratic variation, and are 1 apiece, so that the sum of the squared variations is 2. (A)

**64. [Lesson 18]** The easiest way to work with powers is to log the process. We know that when logging the process,  $0.5\sigma^2$  is subtracted from the  $dt$  term (see Example 18C).

$$d(\ln Y(t)) = (1.2 - 0.5(0.5^2))dt - 0.5dZ(t) = 1.075dt - 0.5dZ(t)$$

Since  $S(t)$  is a stock price, it must be positive, so we take the positive square root of  $Y(t)$ , or equivalently,  $\ln S(t)$  is one-half of  $\ln Y(t)$ :

$$d(\ln S(t)) = 0.5375dt - 0.25dZ(t)$$

For the upper bound of the 90% lognormal confidence interval, we add  $1.645\sigma$  to  $\mu$ , and  $\ln S(0) = \ln 8$ , so the upper bound of the confidence interval for  $\ln S(t)$  is  $\ln 8 + 2(0.5375) + 1.645(0.25)\sqrt{2}$ . Exponentiating, we get  $8e^{2(0.5375)+1.645(0.25)\sqrt{2}} = \boxed{41.93}$ . (C)

**65. [Section 10.2]** It may help to statements (iii)–(vii) in symbols:

(iii)  $\alpha - \delta - 0.5\sigma^2 = 0.10$

(iv)  $2(r - \delta - 0.5\sigma^2) = 0.06$ , or  $r - \delta - 0.5\sigma^2 = 0.03$

(v)  $r = 0.04$

(vi)  $P = 10$

(vii)  $|\Delta_{\text{put}}| = 20$

In (vii), note the use of absolute value signs. The question is intentionally ambiguous, since you are expected to figure out the sign by yourself. When selling a put, you must buy stock to delta hedge, since  $\Delta$  is negative. So  $S\Delta = -20$ .

We are being asked for a time-0 rate, not a rate over a period of time, so the concepts of Section 10.2, rather than the concepts of Lesson 5, are appropriate. In that section, the formula on page 209 for the return on an option sets the risk premium of an option equal to elasticity of option times risk premium of stock:

$$\gamma - r = \Omega(\alpha - r) \tag{10.7}$$

We are given enough information to compute the elasticity, which is  $S\Delta/P$  (see formula (10.5)):

$$\Omega = \frac{S\Delta}{P} = \frac{-20}{10} = -2$$

and we are given  $r = 0.04$ . By subtracting (iv) from (iii), we get  $\alpha - r = 0.10 - 0.03 = 0.07$ . Therefore  $\gamma$  is

$$\gamma = r + \Omega(\alpha - r) = 0.03 - 2(0.07) = -0.10$$

and the absolute value of the rate of return is 10%. (C)

**66. [Lesson 20]** This question tests your ability to add or subtract  $0.5\sigma^2$  when exponentiating or logging a Brownian motion, and your ability to handle negative coefficients for volatility. It is interesting that even in formal exam language, they allow themselves to use a single symbol (like  $X$ ) for both the stock and its price, although an argument is added to indicate the price.

The Sharpe ratios of  $X$  and  $Y$  must be equal, since they're based on a single source of uncertainty. The process for  $X(t)$  is in ideal form for computing the Sharpe ratio. The rate of return is the stock appreciation rate of 0.06 plus the dividend of 0.02, or 0.08, so the Sharpe ratio of  $X$  is

$$\phi_X = \frac{\alpha_X - r}{\sigma_X} = \frac{0.06 + 0.02 - 0.04}{0.2} = 0.2$$

Since  $Y$  is given in exponential form,  $0.5\sigma^2$  must be added to the coefficient of  $t$  to obtain the mean appreciation rate, so  $\alpha - \delta = \mu + 0.5(0.1^2) = \mu + 0.005$ . We are given  $\delta_Y = 0.01$ . *The negative sign on the coefficient of  $Z(t)$  must be maintained when computing the Sharpe ratio.* Thus

$$\phi_Y = \frac{\alpha_Y - r}{\sigma_Y} = \frac{\mu + 0.005 + 0.01 - 0.04}{-0.1} = \frac{\mu - 0.025}{-0.1}$$

Setting this equal to  $\phi_X = 0.2$ , we get  $\mu - 0.025 = -0.02$  and  $\mu = 0.005$ . (A)

Note: Incorrect answer (C) is if you forgot the dividend on  $Y$ . Incorrect answer (E) is if you used 0.1 instead of  $-0.1$  as the denominator of  $Y$ 's Sharpe ratio.

**67. [Lesson 20]** This question tests your ability to add or subtract  $0.5\sigma^2$  when exponentiating or logging a Brownian motion. Since two processes with the same  $dZ(t)$  are given, it is obvious that we're going to equate the Sharpe ratios.

The Sharpe ratio for  $S_2$  is obtained by exponentiating the process. In other words, it is necessary to add  $0.5\sigma^2$  to the drift of the process for  $\ln S_2(t)$ . Also, the dividends must be added to the coefficient of  $dt$  to obtain the total return. I'll use subscripts of 1 and 2 for the parameters of the  $S_1$  and  $S_2$  respectively.

$$\phi_2 = \frac{\alpha_2 - r}{\sigma_2} = \frac{0.03 + 0.5(0.2^2) + 0.01 - 0.04}{0.2} = 0.1$$

The Sharpe ratio of  $S_1$  is

$$\phi_1 = \frac{\alpha_1 - r}{\sigma_1} = \frac{\mu - 0.04}{20\mu}$$

Equating the two Sharpe ratios,

$$\frac{\mu - 0.04}{20\mu} = 0.1$$

$$\mu - 0.04 = 2\mu$$

$$\mu = \text{span style="border: 1px solid black; padding: 2px;">-0.04}$$
 (A)

**68. [Lesson 23]** This question is a clever way of testing you on the popular Ornstein-Uhlenbeck integral. Using the last two lines of Table 23.1, we see that the stochastic differential equation is for an Ornstein-Uhlenbeck process with  $\alpha = 0$ ,  $\lambda = 3$ ,  $\sigma = 2$ . Comparing the integral for the process to the solution we are given, we see that  $e^{-\lambda t}$  can be factored out of the integral in Table 23.1 to obtain (leaving out the  $\alpha$  term, since  $\alpha = 0$  in our case)

$$X(t) = X(0)e^{-\lambda t} + e^{-\lambda t} \sigma \int_0^t e^{\lambda s} dZ(s)$$

Thus  $A = D = \lambda = 3$ ,  $C = \sigma = 2$ , and  $A + C + D = 3 + 2 + 3 = \boxed{8}$ . **(D)**

Alternatively, we can integrate using integrating factors. Move  $3X(t)dt$  to the left side. As discussed in Section 23.3, the integrating factor is  $e^{\int 3dt} = e^{3t}$ . Multiplying through by  $e^{3t}$ , we get

$$\begin{aligned} d(e^{3t} X(t)) &= 2e^{3t} dZ(t) \\ e^{3t} X(t) - X(0) &= 2 \int_0^t e^{3s} dZ(s) \\ e^{3t} X(t) &= X(0) + 2 \int_0^t e^{3s} dZ(s) \\ X(t) &= e^{-3t} \left( X(0) + 2 \int_0^t e^{3s} dZ(s) \right) \end{aligned}$$

and we see that  $A = 3$ ,  $C = 2$ , and  $D = 3$ .

**69. [Section 12.3]** First we must compute the values of the put option at the nodes. Due to the high interest rate, it is likely that early exercise will be optimal.

The up and down ratios are  $u = 1.25$  and  $d = 0.8$ , as we deduce from the stock values. For example,  $150/120 = 1.25$  and  $96/120 = 0.8$ . Then

$$p^* = \frac{e^{0.1} - 0.8}{1.25 - 0.8} = 0.678158$$

The option pays off at the bottom two nodes at the end, with payments of 24 and 58.56 respectively.

At time 2, the value of the put at the upper node is 0. The middle node is

$$P_{ud} = e^{-0.1}(1 - 0.678158)(24) = 6.989161$$

while at the bottom node,

$$P_{dd}^{\text{tentative}} = e^{-0.1}(0.678158(24) + (1 - 0.678158)(58.56)) = 31.7805$$

but exercise of the option at that node yields  $120 - 76.9 = 43.1$ , so  $P_{dd} = 43.1$ .

At time 1, the value of the put at the upper node is

$$P_u = e^{-0.1}(1 - 0.678158)(6.989161) = 2.035349$$

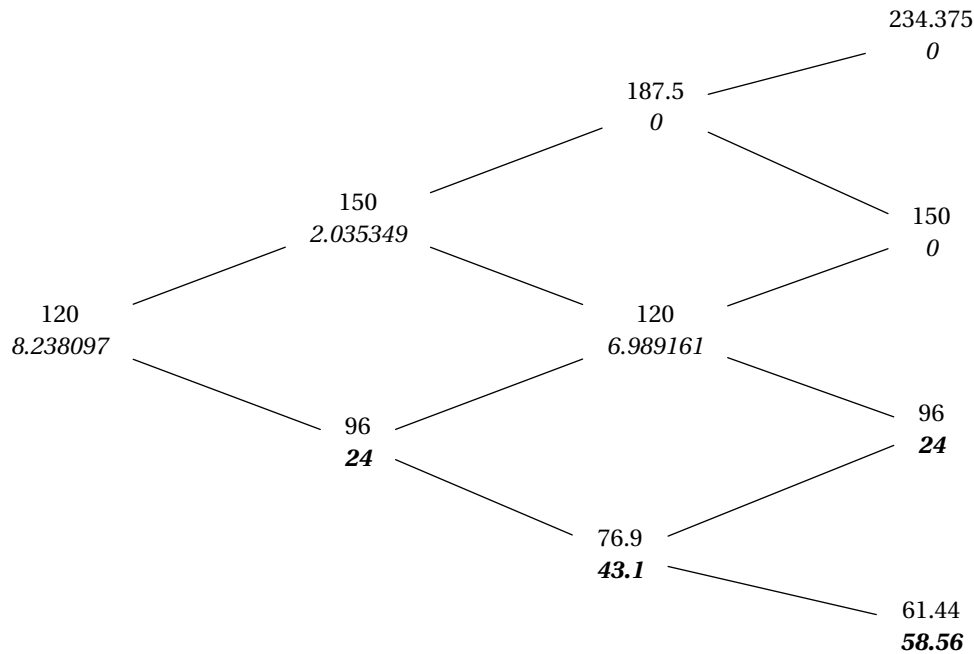
while at the lower node,

$$P_d^{\text{tentative}} = e^{-0.1}(0.678158(6.989161) + (1 - 0.678158)(43.1)) = 16.8401$$

but exercise of the option at that node yields  $120 - 96 = 24$ , so  $P_d = 24$ .

At time 0, the value of the put is

$$P = e^{-0.1}(0.678158(2.035349) + (1 - 0.678158)(24)) = 8.238097$$



**Figure B.8:** Binomial tree with put option values for Sample Question 69

The tree is shown in Figure B.8.

We are now ready to calculate theta. Since  $\varepsilon = 0$  ( $\varepsilon = S_{ud} - S_0$ , and  $S_{ud} = S_0$ ), it is not necessary to calculate delta and theta, and formula (12.6) simplifies to

$$\theta(S, 0) = \frac{C(S_{ud}, 2h) - C(S, 0)}{2h} = \frac{6.989161 - 8.238097}{2} = \boxed{-0.62447} \quad (\text{A})$$

**70. [Section 21.2]** The volatility for a proportional portfolio is the proportion times the volatility of the risky asset, or  $0.8(0.2) = 0.16$ . All the answer choices have this as the volatility.

The Sharpe ratio of the proportional portfolio must equal the Sharpe ratio of the risky asset. The Sharpe ratio of the stock is

$$\phi = \frac{\alpha - r}{\sigma} = \frac{0.1 + 0.02 - 0.05}{0.2} = 0.35$$

Therefore, we back out the  $\alpha$  for the proportional portfolio from

$$\frac{\alpha_p - 0.05}{0.16} = 0.35$$

$$\alpha_p = (0.35)(0.16) + 0.05 = 0.106$$

To get it into the form of the answer choices, in which an arithmetic Brownian motion is exponentiated, we must subtract  $0.5\sigma_p^2$  from  $\alpha_p$ :  $0.106 - 0.5(0.16^2) = 0.0932$ . We see that (C) is the correct answer.

**71. [Section 21.1]** This question is a general question involving risk-neutral pricing. The basic idea is that the value of the derivative security may be computed using the risk-neutral expected value of the payoff, discounted at the risk-free rate. We will use  $\mathbf{E}^*$  to indicate a risk-neutral expectation.

$S(1)$  is a lognormal random variable. The risk-neutral  $\nu$  of the associated normal random variable is 0.40 (from (iii)). As usual, we subtract  $0.5(0.40^2) = 0.08$  from the coefficient of  $dt$  to obtain the  $m$  parameter of the

associated normal random variable;  $m = 0.08 - 0.08 = 0$ . So  $\ln S(1)$  is normal with  $m = 0$  and  $v = 0.4$ , or  $0.4Z$ , where  $Z$  is standard normal. Then

$$\mathbf{E}^* \left[ 1 + S(1) \left( \ln(S(1)) \right)^2 \right] = 1 + \mathbf{E} \left[ e^{0.4Z} (0.4Z)^2 \right] = 1 + 0.16(1 + 0.4^2)e^{0.08}$$

by the formula provided in (v).

The risk-free rate is the rate earned by the stock in the risk-neutral process. It is 0.08, the coefficient of  $dt$ , plus the dividend rate of 0.04, so  $r = 0.08 + 0.04 = 0.12$ . Therefore, the time-0 price of the derivative security is  $e^{-0.12} (1 + 0.16(1.16)e^{0.08}) = \mathbf{1.065}$ . (C)

**72. [Section 14.2 and Lesson 22]** If you buy a gap call and sell a gap put, you are guaranteed to pay the strike price and receive the asset. (There's an ambiguous case if the ending stock price equals the trigger price, but the probability of that in a continuous process is 0, so we can ignore it.) So the present value of a gap call minus a gap put is the prepaid forward value of the stock minus the strike price, or

$$C - P = F^P(S^2) - Ke^{-rt}$$

where  $K$  is the strike price. In our case,  $r = 0.07$  and  $t = 0.5$ . Also, we're given  $C - P = 5.543 + 4.745 = 10.288$ . So

$$10.288 = F^P(S^2) - 95e^{-0.035} \quad (*)$$

It remains to evaluate  $F^P(S^2)$ . Use equation (22.3),

$$\begin{aligned} F^P(S^2) &= e^{-rT} S(0)^2 e^{[2(r-\delta)+0.5(2)(1)\sigma^2]T} \\ &= e^{-0.035} (100)e^{[2(0.07-\delta)+0.01](0.5)} \\ &= 100e^{0.04-\delta} \end{aligned}$$

Substituting into (\*),

$$\begin{aligned} 100e^{0.04-\delta} - 95e^{-0.035} &= 10.288 \\ e^{0.04-\delta} &= \frac{10.288 + 95e^{-0.035}}{100} = 1.020205 \\ 0.04 - \delta &= \ln 1.020205 = 0.02 \\ \delta &= \mathbf{0.02} \quad (\mathbf{A}) \end{aligned}$$

**73. [Lesson 22]** The Itô process for  $S^a$  is given by equation (22.4):

$$\frac{dC}{C} = (a(\alpha - \delta) + 0.5a(a - 1)\sigma^2)dt + a\sigma dZ(t)$$

where  $C = S^a$ . In our question,  $\alpha - \delta = 0.3$  and  $\sigma$  is negated. Equating coefficients of  $dt$  and  $dZ(t)$  to (ii), we get two equations in  $a$  and  $\sigma$ :

$$\begin{aligned} 0.3a + 0.5a(a - 1)\sigma^2 &= -0.66 \\ -a\sigma &= 0.6 \end{aligned}$$

Substituting  $\sigma = -0.6/a$  into the first equation, we get

$$0.3a + \frac{0.5(a - 1)(0.6^2)}{a} = -0.66$$

$$\begin{aligned} 0.3a + 0.18 - \frac{0.18}{a} &= -0.66 \\ 0.3a + 0.84 - \frac{0.18}{a} &= 0 \\ 0.3a^2 + 0.84a - 0.18 &= 0 \end{aligned}$$

Solving the quadratic equation, we get

$$a = \frac{-0.84 \pm \sqrt{0.9216}}{0.6} = 0.2, -3$$

But 0.2 is rejected because it leads to a negative  $\sigma$ . Therefore,  $\sigma = -0.6 / -3 = \boxed{0.2}$ . (B)

**74. [Section 7.2]** We are being asked for the 1st percentile of the payoff of the put option, discounted at 2%. By the structure of the question, we calculate the percentile using true probabilities, but then discount at the risk-free rate, a combination which ordinarily is considered contradictory.

The mean of the associated normal random variable for the 4-year price appreciation is

$$m = (\alpha - 0.5\sigma^2)(t) = (0.1 - 0.5(0.3^2))(4) = 0.22$$

and the standard deviation is

$$v = \sigma\sqrt{t} = 0.3\sqrt{4} = 0.6$$

so the 1st percentile of the variable is  $0.22 - 2.32635(0.6) = -1.17581$  and the 1st percentile of the stock price is  $40e^{-1.17581} = 12.34276$ . Then the put pays  $40 - 12.34276 = 27.65724$ , and discounting this at 2%,  $27.65724e^{-0.08} =$

$\boxed{25.53}$ . (E)

**75. [Section 15.4]** The Boyle modification minimizes variance. As indicated in formula (15.5) on page 351, the resulting variance is the naive variance times the complement of the square of the correlation coefficient, or  $25(1 - 0.8^2) = \boxed{9}$ . (D)

**76. [Section 24.2.2]** We need to fill in the missing interest rate at the middle node of the last period. In a BDT tree, the rates for any period follow a geometric progression, so the middle rate is the geometric average of the other two:  $r_{ud} = \sqrt{(0.172)(0.106)} = 0.135$ , where we round it to three places to be consistent with the official solution. (Without rounding, the final answer would be 0.09 higher.)

We then perform backwards induction. The cap is worthless at the bottom node  $uu$  of the last period, and at the other nodes it is worth: ( $C_x$  is the value of the cap at node  $x$ .)

$$\begin{aligned} C_{uu} &= \frac{10,000(0.172 - 0.115)}{1.172} = 486.348 \\ C_{ud} &= \frac{10,000(0.135 - 0.115)}{1.135} = 176.211 \end{aligned}$$

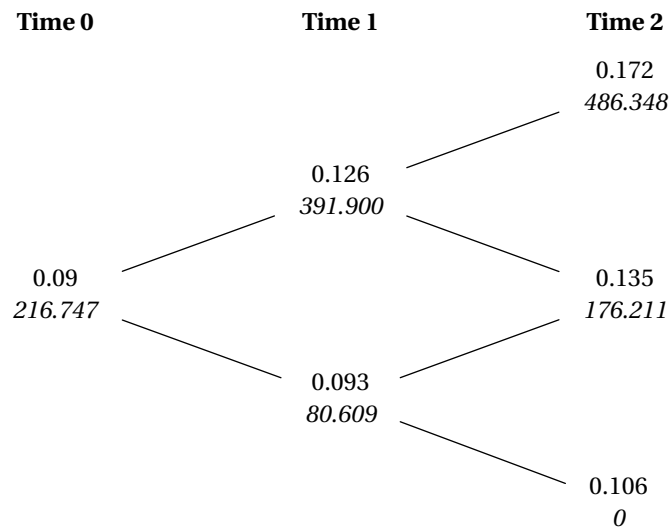
At the end of one year, there is a cap payment at the top node but not at the bottom node. When not stated otherwise, a Black-Derman-Toy probability is assumed to have a 0.5 probability of an up move.

$$\begin{aligned} C_u &= \frac{10,000(0.126 - 0.115) + (0.5)(486.348 + 176.211)}{1.126} = 391.900 \\ C_d &= \frac{0.5(176.211)}{1.093} = 80.609 \end{aligned}$$

The price of the cap is

$$C = \frac{0.5(391.900 + 80.609)}{1.09} = \boxed{216.75} \quad (\text{D})$$

The BDT tree with cap prices is shown in Figure B.9.



**Figure B.9:** Black-Derman-Toy tree with cap values for Sample Question 76